Calc Medic Important Ideas for Unit 1: Intro to Calculus

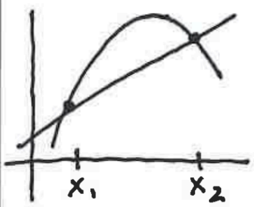
Introducing Calculus: Can Change Occur at an Instant? (Activity: A Wonder-fuel Intro to Calculus)

- Interpret the rate of change at an instant in terms of average rates of change over intervals containing that instant.

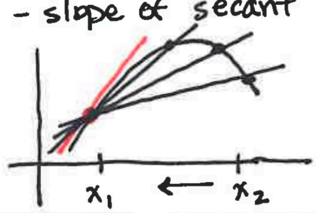
Important Ideas:

Average rate of change vs **Instantaneous rate of change**

Average rate of change
- over a measurable interval
- $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$
- slope of secant line between 2 distinct points



Instantaneous rate of change
- at a moment or instant in time
- $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$
- $\frac{f(x+h) - f(x)}{(x+h) - x}$ as $h \to 0$
- slope of secant line as $x_2 \to x_1$



Defining Limits and Using Limit Notation (Activity: Can You Shoot Free Throws Like Nash?)

- Represent and interpret limits analytically using correct notation, including one-sided limits
- Estimate limits of functions using graphs or tables

Important Ideas: A limit is an intended y-value.

Limit notation has 3 parts:

$\lim_{x \to a} f(x)$ ← function

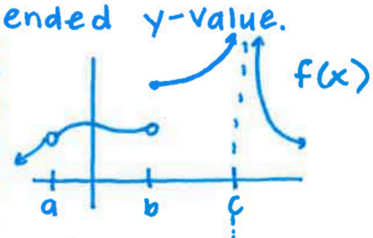
"x approaches a"

What does it mean for a limit to exist?

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \Leftrightarrow \lim_{x \to a} f(x) = L$

* Limits from left + right must match!

* L is a finite #



$\lim_{x \to a} f(x)$ exists bc $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

$\lim_{x \to b} f(x)$ DNE because $\lim_{x \to b^-} f(x) \neq \lim_{x \to b^+} f(x)$

$\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ DNE because $f(x) \to \infty$

Using Algebraic Approaches to Evaluate Limits (Activity: Contestants, Can You Solve This Limit?)

- Use limit properties to determine the limits of functions
- Use algebraic manipulations to determine the limits of functions

Important Ideas:

Strategies to evaluate limits:

- direct substitution (try first)
- memorized forms ($\frac{\sin x}{x}$ or $\frac{|x|}{x}$ ...)
- factor, simplify, substitute
- algebraic manipulation (mult by a form of 1)
- graphs or tables
- limit properties to simplify functions

① $\lim_{x \to a} c = c$ ② $\lim_{x \to a} c f(x) = c \lim_{x \to a} f(x)$ ③ $\lim_{x \to a} \left(f(x) \pm g(x)\right) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

④ $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$ ⑤ $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ only if this lim $\neq 0$

$\frac{k}{0} = +\infty,\ k \neq 0,\ k > 0$; $-\infty,\ k \neq 0,\ k < 0$

$\frac{0}{k} = 0,\ k \neq 0$

$\frac{0}{0}, \frac{\infty}{\infty}$ are indeterminate!

$\frac{k}{\infty} = 0$

Introduction to Squeeze Theorem (Activity: How Many Coffee Beans Are In The Jar?)

- Develop an understanding of bounding values and bounding functions
- Confirm the hypotheses of the Squeeze Theorem (Sandwich Theorem, Pinching Theorem, etc.) and use the theorem to justify a limit result

Important Ideas:

Conditions: If $f(x)$, $g(x)$ and $h(x)$ are continuous functions on some interval containing a, and $g(x) \le f(x) \le h(x)$ on that interval,

Conclusion: then if $\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$, then by the Squeeze Theorem $\lim_{x \to a} f(x) = L$

How to justify w/ Squeeze Thm

① Verify both conditions

② Identify upper bound + lower bound functions

③ Evaluate limits of upper + lower bound functions

④ Make conclusion about original function's limit using the Squeeze Theorem

Continuity and Discontinuity (Activity: Soul Mates at Starbucks)

- Justify conclusions about continuity at a point using the definition.
- Determine intervals over which a function is continuous.

Important Ideas:

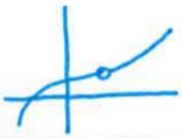
A function $f(x)$ is continuous at $x=a$ if

① $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$ (finite #) "left and right limits match"

② $\lim_{x\to a} f(x) = f(a)$ "limit matches y-value"

Types of Discontinuity

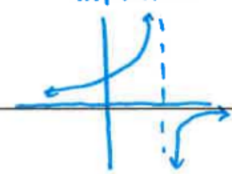
Removable



Non-removable

Jump

Infinite



Removing Discontinuities (Activity: Can This Date Be Salvaged?)

- Determine locations of removable discontinuities by graphical, numeric, or analytic methods
- Determine when and how discontinuous functions can be made continuous

Important Ideas:

Discontinuities that occur where a limit exists can be removed by defining or redefining a point on the graph. ("patch the hole")

1) Evaluate $\lim_{x\to a} f(x)$

2) If the limit exists, define $f(a)$ to be $\lim_{x\to a} f(x)$

* This creates an extended function.

Limits Involving Infinity (Activity: How Much Do We Remember From School?)

- Interpret the behavior of functions using limits involving infinity.

Important Ideas: $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are asking about end behavior (horizontal asymptotes)

For rational functions:

If degree of numerator > degree of denominator, $\lim_{x\to\infty} f(x)$ DNE because $y \to \infty$

horizontal asymptotes

If degree of num < degree of denom, $\lim_{x\to\infty} f(x) = 0$

If degree of num = degree of denominator, $\lim_{x\to\infty} f(x) = \frac{\text{leading coefficient of num}}{\text{leading coefficient of denom}}$

For non-rational functions, compare dominant behavior using TFEPLC.

Tower > Factorial > Exponential > Polynomial > Logarithmic > Constant

Intermediate Value Theorem (Activity: Are You A 5-Star Uber Driver)

- Explain the behavior of a function on an interval using the Intermediate Value Theorem.

Important Ideas: * IVT is used to prove that a certain y-value MUST exist.

Condition: If a function f(x) is continuous on [a, b]

Conclusion: then f(x) attains or "hits" every y-value between f(a) and f(b).

① Verify condition of continuity

② Identify f(a) and f(b)

③ Check that desired value is between f(a) and f(b)

④ Make conclusion using the IVT, incorporating the question stem

Since $\underset{\text{function}}{\underline{\hspace{2cm}}}$ is continuous on [a, b] and f(a) = ____ and f(b) = ____, the IVT guarantees that...

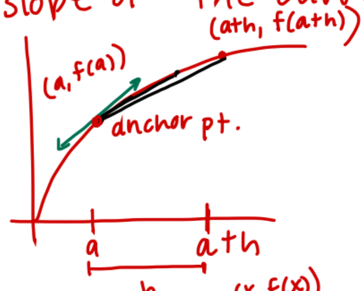
Calc Medic Important Ideas for Unit 2: Differentiation

Instantaneous Rate of Change (Activity: Can a Human Break the Sound Barrier?)

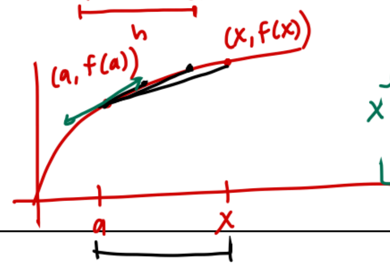
- Determine average rates of change using difference quotients
- Represent the derivative of a function as the limit of a difference quotient

Important Ideas:

Instantaneous rate of change at $x=a$ represents the slope of the curve at $x=a$.



$$\underbrace{\lim_{h\to 0} \underbrace{\frac{f(a+h)-f(a)}{h}}_{\text{avg. R.O.C}}}_{\text{instantaneous R.O.C.}}$$ Definition 1



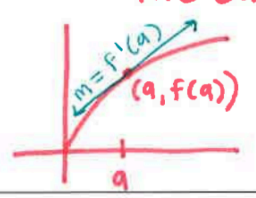
$$\underbrace{\lim_{x\to a} \underbrace{\frac{f(x)-f(a)}{x-a}}_{\text{avg. R.O.C.}}}_{\text{instantaneous R.O.C.}}$$ Definition 2

Defining the Derivative (Activity: The Making of a Slopes Graph)

- Understand that the derivative is itself a function that outputs the slope of the curve at *any* point on the original function.

Important Ideas:

1. A derivative is a function, $f'(x)$, that gives the slope of the curve at any x-value on $f(x)$.
2. Notation for derivative:
   - y'
   - $\frac{dy}{dx}$
   - $\frac{d}{dx}y$
   - $f'(x)$
   - $\frac{df}{dx}$
3. A tangent line touches the curve at one point and shares the slope of the curve.



Equation of tangent line

$y-f(a)=f'(a)(x-a)$

CALC MEDIC

Continuity and Differentiability (Activity: Is This Rollercoaster Safe to Ride?)

- Estimate the derivative at a point using graphs or tables
- Explain the relationship between differentiability and continuity
- Justify how a continuous function may fail to be differentiable at a point in its domain

Important Ideas:

Differentiable functions @ $x=a$ must satisfy both conditions:

I. $f(x)$ is continuous at $x=a$

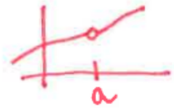
AND

II. $\lim_{x \to a^-} f'(x) = \lim_{x \to a^+} f'(x)$

* slopes must match from both sides!

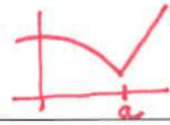
Non-differential functions at $x=a$:

* $f(x)$ not continuous



OR

* slopes from both sides do <u>not</u> match



Continuity <u>does not</u> imply differentiability

Derivative Shortcuts (Activity: Is There a Shortcut?)

- Calculate derivatives of familiar functions

Important Ideas:

Deriv of a constant: $\frac{d}{dx} c = 0$

Deriv of a const. multiplier: $\frac{d}{dx} c\, f(x) = c \frac{d}{dx} f(x) = c \cdot f'(x)$

Power Rule: $\frac{d}{dx} x^n = n x^{n-1}$ for $n \neq 0$

Deriv of Sum or Difference: $\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$
$= f'(x) \pm g'(x)$

Derivatives of Sin x and Cos x (Activity: Toothpick Tangents)

- Calculate the derivatives of sin x and cos

Important Ideas:

If $f(x) = \sin x$, then $f'(x) = \cos x$

If $g(x) = \cos x$, then $g'(x) = -\sin x$

Derivatives of e^x and ln x (Activity: Toothpick Tangents (Part 2))

- Calculate derivatives of familiar functions

Important Ideas:

If $f(x)=e^x$
then $f'(x)=e^x$

If $g(x)=\ln x$
then $g'(x)=1/x$
"the reciprocal of x"

The Product Rule (Activity: How Fast is Snapchat?)

- Calculate derivatives of products of differentiable functions
- Use the product rule in association with other derivative rules

Important Ideas:

Let $h(x)=f(x)g(x)$
then $h'(x)=f(x)g'(x)+g(x)f'(x)$

* Some products can be simplified to avoid using the product rule

* Some functions need to be rewritten so they appear as a product of 2 functions.

Using the Quotient Rule (Activity: Divide and Conquer)

- Use the quotient rule to find derivatives of quotients of differentiable functions
- Simplify the differentiation process by choosing the correct derivative rules

Important Ideas:

For differentiable functions $f(x)$ and $g(x)$, and $g(x)\neq 0$,

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{(g(x))^2}$$

Check results with Math 8
Check results by comparison to Product Rule
Simplify original rational function, if possible

Derivatives of Trig Functions (Activity: Tangents for Trig Functions)

- Calculate derivatives of products of differentiable functions
- Use identities to rewrite tangent, cotangent, secant, and cosecant functions and then apply derivative rules to find formulas for their derivatives
- Use the rules for derivatives of trigonometric functions in association with other derivative rules

Important Ideas:

$\frac{d}{dx}\sin x = \cos x$ $\quad \frac{d}{dx}\tan x = \sec^2 x$ $\quad \frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\cos x = -\sin x$ $\quad \frac{d}{dx}\cot x = -\csc^2 x$ $\quad \frac{d}{dx}\csc x = -\csc x \cot x$

Memorize these formulas!

Simplify or rewrite original functions, if possible

Calc Medic Important Ideas for Unit 3
Differentiating Composite, Implicit, and Inverse Functions

The Chain Rule (Activity: How is Lindt Chocolate Made?)

- Calculate derivatives of compositions of differentiable functions

Important Ideas:

Composite Functions

outside function

$y = f(g(x))$

inside function

"■"

Chain Rule is used for finding derivatives of composite functions.

$y' = g'(x) \cdot f'(g(x))$

$\frac{dy}{dx} = \frac{d■}{dx} \cdot \frac{dy}{d■}$

Implicit Differentiation (Activity: The Tangent Line Problem (Revisited))

- Find the derivative of implicitly defined functions

Important Ideas:

1. Implicit functions are those where the dependent variable (y) is not isolated on one side of the equation. Ex: $x^2+xy-y^2=1$
2. Steps for differentiating an inverse function:
   1) Differentiate both sides w/ respect to x.
   2) Apply chain rule to all terms with y in them
   3) Collect all terms w/ $\frac{dy}{dx}$ on one side of the equation
   4) Factor out $\frac{dy}{dx}$
   5) Solve for $\frac{dy}{dx}$ by dividing
3. To find $\frac{d^2y}{dx^2}$, the 2nd derivative, repeat the process + substitute the function for $\frac{dy}{dx}$

Derivatives of Inverse Functions (Activity: What's Your Slope?)

- Calculate derivatives of inverse functions

Important Ideas:

An inverse function is a function that reverses or "undoes" another function. If $f(a)=b$, then $f^{-1}(b)=a$.

An inverse function exists if the original function is one-to-one. (passes Horizontal Line Test)

*Slopes at inverse points are reciprocals!

If (a,b) is on the graph of f, and g is the inverse of f, then

$$g'(b)=\frac{1}{f'(a)}$$

In general:

$$\frac{d}{dx}\left(g(x)\right)=\frac{1}{f'(g(x))}$$

the reciprocal of

the derivative of f

evaluated at the inverse pt.

Derivatives of Inverse Trigonometric Functions (Activity: Getting Triggy With It)

- Calculate derivatives of inverse trig functions

Important Ideas:

$$\frac{d}{dx}(\arcsin x)=\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x)=\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x)=\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

Calc Medic Important Ideas for Unit 4: Contextual Applications of Differentiation

Interpreting the Meaning of the Derivative in Context (Activity: A Summer Day of Calculus)

- Interpret the meaning of a derivative in context.

Important Ideas:

- The derivative represents the rate of change of the dependent variable with respect to the independent variable. $f'(x) = \frac{df}{dx}$
- Units of $f'(x) = \frac{\text{units of } f(x)}{\text{units of } x}$
- Units of $f''(x) = \frac{\text{units of } f'(x)}{\text{units of } x}$
- Interpreting $y'(x)$

"At $x =$ ____, the y-context is increasing/decreasing at a rate of ____ (correct units)"

Connecting Position, Velocity and Acceleration (Activity: The Lovely Ladybug)

- Calculate rates of change in the context of straight-line motion.

Important Ideas:

1 Position $= s(t)$

Velocity = speed w/ direction

$v(t) = s'(t)$

Acceleration

$a(t) = v'(t) = s''(t)$

2 $v(t) = 0 \Rightarrow$ object is at rest

$v(t) > 0 \Rightarrow$ object is moving right or up

$v(t) < 0 \Rightarrow$ object is moving left or down

3

	$a(t) > 0$	$a(t) < 0$
$v(t) > 0$	speeding up	slowing down
$v(t) < 0$	slowing down	speeding up

Compare sign of velocity & acceleration

Rates of Change in Applied Contexts Other than Motion (Activity: How Many Shoppers on Black Friday?)

- Interpret rates of change in applied contexts.

Important Ideas:

$y'(t)$ is the rate of change of $y(t)$.

$y'(t) = 0 \Rightarrow$ y-context is not changing.

$y'(t) > 0 \Rightarrow$ y-context is increasing.

$y'(t) < 0 \Rightarrow$ y-context is decreasing.

Calculator Tips

1 For rate in/rate out problems:

$Y_1 =$ rate in

$Y_2 =$ rate out

$Y_3 = Y_1 - Y_2$

2 Always round to at least 3 decimal places!

Intro to Related Rates (Activity: Birthday Balloons)

- Calculate and interpret related rates in applied contexts

Important Ideas:

Related Rate Problems

(1) Draw a picture.

(2) Write an equation that relates all the variables in the problem (usually a volume formula or Pythagorean Thm)

(3) Take derivative of both sides
Don't forget chain rule!

(4) Plug in known values + solve for the quantity you are after.

* Determine based on context if the given rates are positive or negative.

Related Rates (Activity: "Coney" Island)

- Calculate and interpret related rates in applied contexts.

Important Ideas:

Cone Problems

1) Relate the radius to the height using similar Δs

2) Write volume equation only in terms of r, or only in terms of h, depending on what info is given/needed

3) Take derivative of both sides and solve for desired quantity

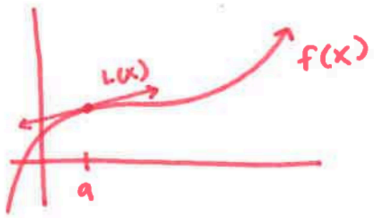
4) Use equation from step 1 to find rate of change of eliminated variable.

Approximating Values of a Function Using Local Linearity and Linearization (Activity: Close Enough is Good Enough!)

- Approximate the value on a curve using the equation of a tangent line

Important Ideas:

Tangent lines can be used to approximate function values near the point of tangency.



For x-values near $x=a$,
$L(x) \approx f(x)$

- If $f(x)$ is concave up, $L(x)$ gives an underestimate. (tangent line is below curve)
- If $f(x)$ is concave down, $L(x)$ gives an overestimate. (tangent line is above curve)

L'Hospital's Rule (Activity: Mixed Messages)

- Interpret the rate of change at an instant in terms of average rates of change over intervals containing that instant.

Important Ideas:

Limits resulting in $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ are considered indeterminate.

Consider the ratio of growth rates instead!

L'Hospital's Rule:

If $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ is indeterminite, evaluate $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$

If the result is still indeterminate, repeat the process by finding another derivative.

Calc Medic Important Ideas for Unit 5: Analytical Applications of Derivatives

The Mean Value Theorem (Activity: Can Calculus Get You Fined?)

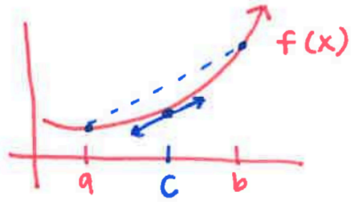
- Justify conclusions about functions by applying the MVT over an interval.

Important Ideas:

Conditions: If a function $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b),

Conclusion: then there exists a value c for $a<c<b$ such that

$$\underbrace{f'(c)}_{\text{instantaneous R.O.C.}} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{average R.O.C.}}$$



How to prove w/ MVT:

1) Check conditions
2) Find average R.O.C. on $[a,b]$
3) Make conclusion using MVT, incorporating the question stem

Extreme Value Theorem, Absolute vs. Relative Extrema, and Critical Points (Activity: What's the Value of Apple Stock?)

- Justify conclusions about functions by applying the Extreme Value Theorem.
- Distinguish between absolute and relative extrema and critical points.

Important Ideas:

1. Extreme Value Theorem:
   If a function $f(x)$ is continuous on $[a,b]$ then $f(x)$ must attain a maximum and minimum value on $[a,b]$.
2. $f(x)$ has an absolute maximum at $x=c$ if $f(c) \geq f(x)$ for ALL x
   $f(x)$ has a relative maximum at $x=c$ if $f(c) \geq f(x)$ for x near c.
   $f(x)$ has an absolute minimum at $x=c$ if $f(c) \leq f(x)$ for ALL x.
   $f(x)$ has a relative minimum at $x=c$ if $f(c) \leq f(x)$ for x near c.
3. Critical points are points where the derivative is 0 or undefined.

Determining Function Behavior from the First Derivative (Activity: Playing the Stock Market)

- Determine behaviors of a function based on the derivative of that function.

Important Ideas:

① $f(x)$ is increasing if $f'(x) > 0$

$f(x)$ is decreasing if $f'(x) < 0$

② First Derivative Test for Relative (Local) Extrema

- Identify critical points ($f'(x) = 0$ or undefined)
- Make a labeled sign chart by testing values between the critical values to determine if f' is pos or neg
- Make conclusion

"$f(x)$ has a relative max of ___ at $x =$ ___ because f' changes from positive to negative."

"$f(x)$ has a relative min of ___ at $x =$ ___ because f' changes from negative to positive"

Using the Candidates Test to Determine Absolute Extrema (Activity: Are You a Stock Market Master?)

- Justify conclusions about the behavior of function based on its derivative.

Important Ideas:

Finding absolute (global) maxima + minima

1) Find and list all the critical values and end points
2) Compare the values of the function at all these locations (Make a table!)

x	f(x)

3) Write conclusion

"$f(x)$ has an absolute max of ___ at $x =$ ___."

"$f(x)$ has an absolute min of ___ at $x =$ ___."

* Must refer to Candidates Test in justification

Analyzing Function Behavior with the Second Derivative (Activity: How Fast Does the Flu Spread?)

- Justify conclusions about the behavior of function based on its second derivative

Important Ideas:

$f''(x)$ tells us how $f'(x)$ (the slopes of f) are changing.

$f''(x) > 0 \Rightarrow f'(x)$ is increasing $\Rightarrow f(x)$ is concave up

$f''(x) < 0 \Rightarrow f'(x)$ is decreasing $\Rightarrow f(x)$ is concave down

$f''(x) = 0$ or undefined AND changes sign $\Rightarrow f'(x)$ has a rel max or min $\Rightarrow f(x)$ has a pt. of inflection

2nd derivative test (using concavity to determine max or min)

- $f(x)$ has a rel. max at $x = c$ if $f'(c) = 0$ and $f''(x) < 0$ max
- $f(x)$ has a rel. min at $x = c$ if $f'(c) = 0$ and $f''(x) > 0$ min

Optimization (Activity: Canalysis)

- Use derivatives to solve optimization problems.
- Interpret maximums and minimums in applied contexts.

Important Ideas:

Optimization is about finding a max or min in applied contexts

① Write an equation for the quantity that is to be maximized or minimized (volume, area, cost, distance, etc.)

② Use the constraints to find relationships between the variables

③ Rewrite your equation with only 1 variable

④ Use the 1st and 2nd derivative tests to find critical values and extrema

Exploring Behaviors of Implicit Relations (Activity: What About Us?)

- Determine critical points of implicit relations.
- Justify conclusions about the behavior of an implicitly defined relation based on evidence from its derivatives.

Important Ideas:

Implicit differentiation can be used to find 1st and 2nd derivatives of relations.

Critical points are the x and y values where $\frac{dy}{dx}=0$ or is undefined.

*Make sure they satisfy the original equation!

A curve is increasing when $\frac{dy}{dx}>0$ and decreasing when $\frac{dy}{dx}<0$. (Specify x and y values!)

A curve is concave up when $\frac{d^2y}{dx^2}>0$ and concave down when $\frac{d^2y}{dx^2}<0$ (specify x and y values!)

Calc Medic Important Ideas for Unit 6: Integration and Accumulation of Change

Exploring Accumulation of Change (Activity: How Much Snow Is On Janet's Driveway?)

- Interpret the meaning of area under a rate of change function in context.

Important Ideas:

1. Accumulated quantity = Rate $1 \cdot \Delta t_1$ + Rate $2 \cdot \Delta t_2$ + ... + Rate $n \cdot \Delta t_n$
2. The accumulation of a quantity is represented by the area underneath its derivative curve.
3. Units for area underneath a rate of change curve: Units of rate of change • Units of independent variable
4. Area above x-axis is positive area ⇒ positive accumulation ⇒ quantity is increasing
   Area below x-axis is negative area ⇒ negative accumulation ⇒ quantity is decreasing

Approximating Areas with Riemann Sums (Activity: Fast and Curious)

- Approximate area under a curve using geometric and numerical methods

Important Ideas:

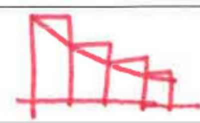
To approximate the area on $[a,b]$ with n equal subdivisions use $\frac{b-a}{n}$ as the width of each rectangle.

To find the height of each rectangle:
- LRAM – Evaluate function at left endpoint of each interval
- RRAM – Evaluate function at right endpoint of each interval
- MRAM – Evaluate function at midpoint of each interval

Total area/accumulation = $\Delta x_1 \cdot f(x_1) + \Delta x_2 f(x_2) + \ldots + \Delta x_n f(x_n)$

If a function is increasing, LRAM gives an underestimate, RRAM gives an overestimate.

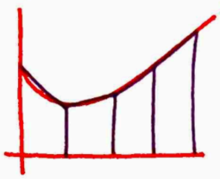
If a function is decreasing, LRAM gives an overestimate, RRAM gives an underestimate.



Approximating Areas with Trapezoids (Activity: Fast and Curious (Part 2))

- Approximate area under a curve using geometric and numerical methods

Important Ideas:



Area of a trapezoid = $\frac{(b_1 + b_2)}{2} \cdot h$

Evaluate function at right endpoint

Evaluate function at left endpoint

Δx

Area under f(x) ≈ sum of trapezoids

A trapezoidal sum gives the <u>average</u> of LRAM and RRAM values.

Riemann Sums, Summation Notation, and Definite Integrals (Activity: How Confident Are You?)

- Interpret and represent an infinite Riemann sum as a definite integral

Important Ideas:

To find <u>exact</u> area under a curve f(x) over the interval [a,b] use infinitely many rectangles of infinitesimally small widths.

Summation Notation:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i)}_{\text{height of rectangle}} \underbrace{\Delta x}_{\text{width of rectangle}}$$

n: # of rectangles

Integral Notation:

$$\int_a^b \underbrace{f(x)}_{\text{integrand}} d\underline{x}$$

b: upper limit of integration

a: lower limit of integration

x: variable of integration

The Fundamental Theorem of Calculus and Accumulation Functions (Activity: Under Cover)

- Represent accumulation functions using definite integrals.
- Find derivatives of accumulation functions.

Important Ideas:

An accumulation function outputs the area under a curve from some starting value to x (the input).

$$F(x) = \int_c^x f(t)\,dt$$

x: independent variable

f(t): rate of accumulation

c: starting value

FTC (Part 1): Rate of change of an accumulation function

$$\frac{d}{dx}\int_c^x f(t)\,dt = \frac{d}{dx}F(x) = f(x)$$

Applying Properties of Definite Integrals (Activity: #2020 Goals)

- Calculate a definite integral using areas and properties of definite integrals.

Important Ideas:

For $a < b < c$

① $\int_a^c f(t)\,dt = \int_a^b f(t)\,dt + \int_b^c f(t)\,dt$

④ $\int_a^b k f(t)\,dt = k\int_a^b f(t)\,dt$

② $\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt$

⑤ $\int_a^a f(t)\,dt = 0$

③ $\int_a^b [f(t) \pm g(t)]\,dt = \int_a^b f(t)\,dt \pm \int_a^b g(t)\,dt$

The FTC and Definite Integrals (Activity: Go Figure)

- Evaluate definite integrals analytically using the Fundamental Theorem of Calculus.

Important Ideas:

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$ $\left(F'(x) = f(x)\right)$

$$\underbrace{F(b)}_{\text{Final value}} = \underbrace{F(a)}_{\text{Initial value}} + \underbrace{\int_a^b f(x)\,dx}_{\text{Accumulated change}}$$

Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation (Activity: A Match Made in Heaven)

- Determine antiderivatives of functions and indefinite integrals, using knowledge of derivatives.

Important Ideas:

① Given a function $f(x)$, the most general antiderivative of $f(x)$ is given by

$$\int f(x)\,dx = F(x) + C \quad \text{where } F'(x) = f(x)$$

indefinite integral (no upper + lower bounds)

constant of integration

② Antiderivative Rules

$\int k\,dx = kx + C$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$

$\int \frac{1}{x}\,dx = \ln|x| + C$

$\int e^x\,dx = e^x + C$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

Integration using Substitution (Activity: Which One Doesn't Belong?)

- Use u-substitution to find antiderivatives of composite functions

Important Ideas:

$$\frac{d}{dx}\left(f(u(x))\right) = u'(x)\cdot f'(u(x))$$

$$\int \underbrace{u'(x)}_{\text{derivative of inner function}} \cdot f'(\underbrace{u(x)}_{\text{inner function}})\,dx = \underbrace{f}_{\text{antiderivative of } f'}(u(x)) + C$$

Integration with substitution reverses the chain rule!

① Set u = inner function

② Find $\frac{du}{dx}$, solve for du.

③ "Correct" the integrand so it matches du exactly. (Can only multiply by a scalar form of 1)

④ Rewrite the entire integral in terms of u.

⑤ Find the antiderivative.

⑥ For indefinite integrals, plug in the expression for u

Riemann Sums, Summation Notation, and Definite Integral Notation (Activity: Returning to Riemann)

- Interpret and represent an infinite Riemann sum as a definite integral

Important Ideas:

A definite integral can be represented by an infinite Riemann sum.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{f(x_i)}_{\text{height of } i^{th} \text{ rectangle}}\underbrace{\Delta x}_{\text{width of rectangle}}$$

where n is the # of partitions/rectangles

Assuming equal partitions:

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{f\left(a + i\left(\frac{b-a}{n}\right)\right)}_{\text{height}} \cdot \underbrace{\frac{b-a}{n}}_{\text{width}}$$

Calc Medic Important Ideas for Unit 7: Differential Equations

Modeling Differential Equations and Verifying Solutions (Activity: How Long Does Coffee Stay Hot?)

- Interpret differential equations given in context
- Verify solutions to differential equations

Important Ideas:
A differential equation is any equation that contains a derivative expression.
$\frac{dy}{dx}$ is a first order derivative, $\frac{d^2y}{dx^2}$ is a second order derivative, ...

Diff eqs. may contain the original function or be written in terms of the independent variable only.

If a rate is proportional to the current quantity then $\frac{dy}{dx} = ky$ where "k" is the constant of proportionality.

Slope Fields (Activity: Seeing is Believing)

- Create slope fields
- Estimate solutions to differential equations

Important Ideas:
Slope fields are a graphical representation of a differential equation that allow us to visualize the family of solution curves.

Making a slope field:
- calculate slope at various ordered pairs
- plot slopes using short line segments

A solution curve will follow the trend of the slopes and must pass through the initial condition if given.

If $\frac{dy}{dx}$ is undefined, do NOT draw a slope there.

Finding General Solutions using Separation of Variables (Activity: Are you a Solution Seeker?)

- Determine general solutions to differential equations

Important Ideas:
To solve a first-order differential equation of the form $\frac{dy}{dx} = f(x)g(y)$, use separation of variables.

1. Separate (Move all terms w/ y to one side and all terms w/ x to the other)
2. Integrate (Don't forget +C!)
3. Isolate (Get y by itself, watch out for ± solutions)

Finding Particular Solutions using Initial Conditions and Separation of Variables (Activity: How many Sea Lions are on Elliott Bay?)

- Determine particular solutions to differential equations

Important Ideas:

① Some particular solutions can't be written explicitly and must be defined by integrals.

If $\frac{dy}{dx} = f(x)$ and $F(a) = y_0$ then

$\underbrace{F(x)}_{\text{antiderivative of } f(x)} = y_0 + \int_a^x f(t)\,dt$ is a solution

② Finding particular solutions requires 2 extra steps

Separate Integrate Solve for C Isolate Select

Remember SISIS!

Exponential Models with Differential Equations (Activity: How Fast is the Coronavirus Spreading?)

- Interpret the meaning of a differential equation and its variables in context

Important Ideas:

$\frac{dy}{dt} = ky \Rightarrow$ "rate of change of a quantity is proportional to the quantity" $\Rightarrow$ exponential growth/decay model

$k > 0 \Rightarrow$ growth

$k < 0 \Rightarrow$ decay

$\frac{dy}{dt} = ky$ has solutions of the form $y = y_0 e^{kt}$

initial condition when $t=0$

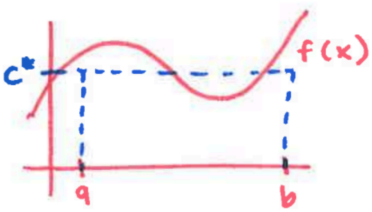
Calc Medic Important Ideas for Unit 8: Applications of Integration

Average Value of a Function (Activity: Finding the Perfect Rectangle)

- Determine the average value of a function using definite integrals

Important Ideas:

The average value of a continuous function f(x) on the interval [a,b] is the height of the rectangle that encompasses the same area as the area under the curve.



$$c^*(b-a) = \int_a^b f(x)\,dx$$

$$c^* = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Connecting Position, Velocity, and Acceleration using Integrals (Activity: Whitney's Bike Ride)

- Determine values for positions and rates of change using definite integrals in problems involving rectilinear motion

Important Ideas:

Given velocity v(t) and position s(t)

$$\int_a^b v(t)\,dt = \underbrace{s(b)-s(a)}_{\substack{\text{change in position} \\ \text{net distance} \\ \text{displacement}}} \quad \text{or} \quad \underbrace{s(b)}_{\substack{\text{Final} \\ \text{position}}} = \underbrace{s(a)}_{\substack{\text{initial} \\ \text{position}}} + \underbrace{\int_a^b v(t)\,dt}_{\substack{\text{change in} \\ \text{position}}}$$

$$\text{Total distance traveled on } [a,b] = \int_a^b |v(t)|\,dt$$

Using Accumulation and Definite Integrals in Applied Contexts (Activity: How many People are at the Met?)

- Interpret the meaning of a definite integral in accumulation problems
- Determine net change using definite integrals in applied contexts

Important Ideas:

Given a quantity $y(t)$ and its rate of change $y'(t)$

① $\int_a^b y'(t)\,dt = y(b) - y(a)$

integral of a rate of change = net change in quantity

② The accumulation equation:

$$y(t) = y(a) + \int_a^t y'(x)\,dx$$

③ Sometimes $y'(t)$ consists of a rate in and a rate out.

Calc tips:

Y_1 = rate in

Y_2 = rate out

$\boxed{Y_3 = Y_1 - Y_2}$

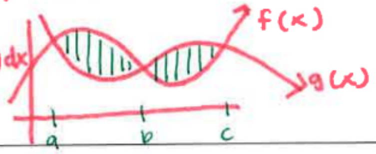
Finding Areas between Curves Expressed as Functions of x (Activity: How Rich are the Top 1%?)

- Calculate areas in the plane using the definite integral

Important Ideas:

- Area between $f(x)$ and $g(x)$ on $[a,b]$ when $f(x) \geq g(x)$ is given by $A = \int_a^b (\underbrace{f(x)}_{\text{upper curve}} - \underbrace{g(x)}_{\text{lower curve}})\,dx$
- The region must be bounded.
- Area is ALWAYS positive.
- Sometimes the upper + lower functions switch!

$$A = \int_a^b (f(x) - g(x))\,dx + \int_b^c (g(x) - f(x))\,dx$$



Finding Areas between Curves Expressed as Functions of y (Activity: How Do You Build a Deck?)

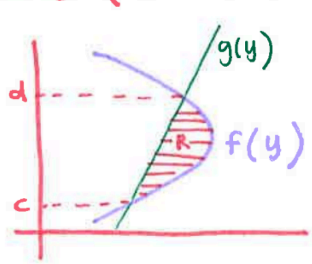
- Calculate areas in the plane using the definite integral

Important Ideas:

If the upper curve requires 2 or more definitions, consider using a right curve and a left curve (horizontal rectangles)

Area of Region R $= \int_c^d (\underbrace{f(y)}_{\text{right curve}} - \underbrace{g(y)}_{\text{left curve}})\,\underbrace{dy}_{\text{width}}$

y-values

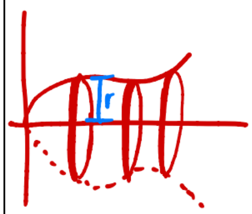


Volume using the Disc Method (Activity: What is the Volume of a Pear?)

- Calculate volumes of solids of revolution using definite integrals

Important Ideas:

Some solids are generated by revolving 2-d regions around an axis of revolution.



If the axis of revolution is a boundary of the region, then each slice is a disk.

$$V=\int_a^b \pi r^2\,dx = \pi\int_a^b r^2\,dx \text{ or } V=\pi\int_c^d r^2\,dy$$

How to find the radius:

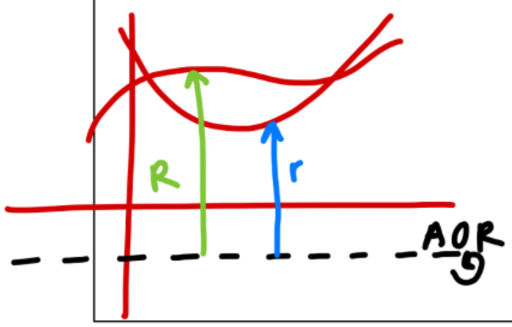
Draw line from AOR to the curve, then "upper-lower" or "further right-closer"

Volume using the Washer Method (Activity: What's the Volume of a Bagel?)

- Calculate volumes of solids of revolution using definite integrals

Important Ideas:

use washers method when there is a space/gap between the region and axis of revolution.



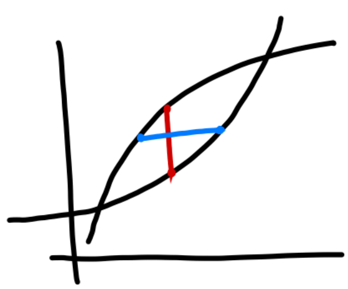
1) Draw sample radii from the axis of revolution to boundaries of region.
2) write an expression for R and r. Think "upper-lower" or "further right-closer"
3) Set-up integral

$$\pi\int_a^b\left[(R(x))^2-(r(x))^2\right]dx \text{ OR } \pi\int_c^d\left[(R(y))^2-(r(y))^2\right]dy$$

Volumes With Cross Sections (Activity: The Best Thing Since Sliced Bread)

- Calculate volumes of solids with known cross sections using definite integrals

Important Ideas:



1) Draw a sample cross-section

2) Write expression for area of one cross-section $A(x)$ or $A(y)$
3) Set-up integral

Perpendicular to x-axis or Perpendicular to y-axis

$$\int_a^b A(x)\,dx \text{ or } \int_c^d A(y)\,dy$$