Calc Medic QuickNotes for Unit 1: Exploring Rates of Change

Functions and Function Notation (Activity: Can We Predict Maximum Heart Rate?)

- Understand that a function describes the relationship between an independent variable and a dependent variable where each input value is mapped to exactly one output value. Functions can be expressed with an equation, table, graph, or verbal description.
- Describe the set of inputs of a function (the domain) and the set of outputs (the range).
- Use and interpret function notation.

QuickNotes

LT # 1: A function describes the relationship between an independent variable and a dependent variable.
Each input is mapped to exactly one output.
To evaluate a function, find the output for a given input using a table, graph, or equation.

LT # 2: The set of inputs of a function is called the <u>domain</u>.
(values where the function is defined)
The set of outputs of a function is called the <u>range</u>.

LT# 3: $\boxed{\underset{\text{function name}}{f}(\overset{\text{input}}{a})} = \overset{\text{output}}{b}$

Interpret Graphs of Functions (Activity: How Does the Food Industry Set Prices?)

- Describe how two quantities vary with respect to each other from a graph in a contextual scenario.
- Determine when a function is increasing or decreasing.
- Interpret key points and graph behavior in context.

QuickNotes

LT # 1: A graph can show how two quantities vary with respect to each other.

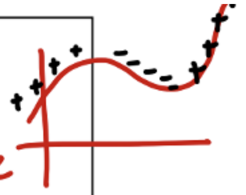
LT#2: A function f is increasing on an interval of its domain if, as the input values increase, the output values increase as well. If $a<b$, then $f(a)<f(b)$
A function f is decreasing on an interval of its domain if, as the input values increase, the output values decrease.
If $a<b$ then $f(b)<f(a)$

LT#3 x-intercepts maximum and minimum values, and intervals of inc/dec can reveal important info about a context.

Concavity (Activity: What is the Business Cycle?)

- Connect the sign of a graph's slope to the increasing or decreasing behavior of a function and the value of the slope to the function's rate of change.
- Use the concavity of a function's graph to describe the change in the function's rate of change and vice versa.

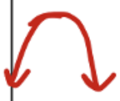
QuickNotes



LT # 1: If a graph has a positive slope then the output is increasing. If a graph has a negative slope the output is decreasing. The value of the slope gives the rate of change.

LT# 2: Rate of change is inc ⇒ slopes are increasing ⇒ graph is concave up (like a cup)

Rate of change is dec ⇒ slopes are decreasing ⇒ graph is concave down (like a frown)



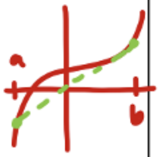
A pt. of inflection is where the concavity changes ⇒ steepest slope

Remember: If slopes are getting more negative, they are decreasing, but the quantity is changing <u>faster</u>.

Rates of Change (Activity: What is Pamela’s Current Speed?)

- Find and interpret a function's average rate of change over an interval.
- Estimate and interpret a function's rate of change at a point.
- Compare rates of change at different intervals or values of a function's domain.

QuickNotes



LT# 1: Average rate of change over an interval [a,b] is $\frac{f(b)-f(a)}{b-a} = \frac{\Delta y}{\Delta x}$

Slope of the straight line between x=a and x=b

“On average, the ________ y-variable changes by ________ avg. rate of change ________ y-unit per unit of ________ x-variable.

LT# 2: The rate of change at a point x=a is given by the slope of the tangent line at x=a and can be estimated by finding an average rate of change on a small interval around x=a.

Change in Linear Functions (Activity: How Much Does It Cost to Rent a U-Haul?)

- Understand that a linear function has a constant rate of change over any interval of its domain.
- Explain why the rate of change of the average rates of change of a linear function is zero.
- Interpret the slope of a linear function in terms of a rate of change.

QuickNotes

LT # 1: Linear functions have a constant rate of change. Each equal interval of the domain produces an equal change in the outputs.

LT # 2: Rate of change is constant, so rate of change of rate of change is zero. (Lines have no concavity)

LT # 3: The slope of a line represents the constant rate of change.

$m > 0$ quantity is increasing

$m < 0$ quantity is decreasing

Change in Quadratic Functions (Activity: How Fast Does a Penny Fall from the Empire State Building?)

- Understand that for quadratic functions, the change in output values over equal intervals of the domain grows linearly.
- Explain why the rate of change of the average rates of change of a quadratic function is constant.
- Connect the concavity of a parabola to whether the average rates of change of the quadratic function are increasing (concave up) or decreasing (concave down).

QuickNotes

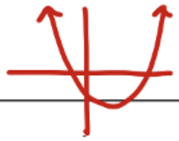
LT # 1 Quadratic functions have a constant second difference, meaning the change over each interval grows or declines linearly.

LT # 2 Average R.O.C. takes the change in output + divides by the change in input, for a "per unit" measure.

- Average rate s of change grow/decline linearly also
  - The change in avg. ROC is constant.

LT# 3

concave down
avg. ROC is decreasing



concave up
avg. ROC is increasing

Calc Medic QuickNotes for Unit 2: Polynomial and Rational Functions

Polynomial Functions and Rates of Change (Activity: Can We Predict Stock Values?)

- Identify key characteristics of a polynomial function including its degree, leading coefficient, relative and absolute extrema, and points of inflection.
- Determine the degree of a polynomial using first, second, third, …nth differences.

QuickNotes LT#1 Polynomial functions can be written in the form

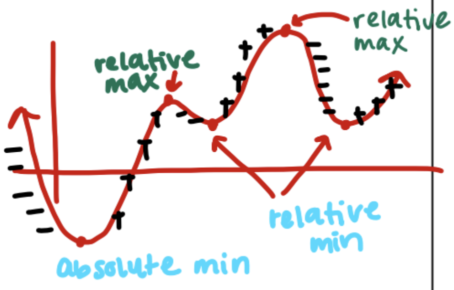
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

n is the degree of the polynomial

a_n is the leading coefficient

Relative extrema occur when a function changes from increasing to decreasing (max) or from decreasing to increasing (min) or at an endpoint (if the domain is restricted).

The absolute extrema are the highest + lowest y-values of the function. *A function may not have an absolute max or min.



LT#2 A polynomial of degree n has a constant n^{th} difference

Zeros of Polynomial Functions (Activity: What's Up With the Zeros?)

- Use a root's multiplicity to describe the polynomial graph's behavior at an x-intercept.
- Understand that a polynomial of degree n has exactly n complex zeros and can be written as a product of n linear factors.
- Find all zeros of a polynomial function when given in factored form; identify when zeros will be imaginary based on the polynomial's graph or equation in factored form.

QuickNotes

LT #1: multiplicity - the # of times a factor occurs

$f(x) = (x-p)^k$

"p is a zero w/ multiplicity k"

k is odd: PASSES THROUGH

k is even: BOUNCE

LT# 2: A polynomial of degree n has exactly n zeros including multiplicity and can be written as $f(x) = a(x-x_1)(x-x_2)(x-x_3)\ldots$ where $x_1, x_2 \ldots x_n$ are the real or complex zeros

LT# 3: If a factor is quadratic, imaginary zeros can be found w/ quadratic formula. Can then write function as a product of n linear factors (fully factored form)

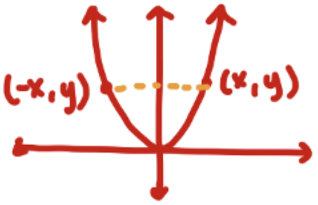
Even and Odd Polynomials (Activity: Seeing Symmetry: Faces and Functions)

- Understand the properties of even and odd functions.
- Algebraically prove whether a polynomial function is even, odd, or neither.

QuickNotes

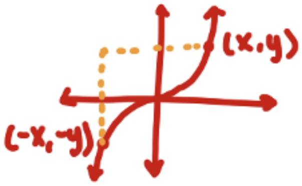
LT#1

A function is <u>even</u> if it has y-axis symmetry.



$f(x)=f(-x)$

A function is <u>odd</u> if it has origin symmetry (180° rotational symmetry)



$f(-x)=-f(x)$

LT#2

To prove symmetry algebraically:
Plug in (-x) for x and simplify expression.
Compare to original.

Polynomial Functions and End Behavior (Activity: The End is in Sight)

- Determine the end behavior of a polynomial from its degree and leading coefficient.
- Explain why the end behavior of a polynomial function is determined by its leading term.
- Use limit notation to describe the end behavior of a polynomial function.

QuickNotes

LT#1 and #2 For very large positive + very large negative input values (at ends of the graph) the polynomial behavior is dominated by the leading term.

		Leading coefficient	
		positive	negative
Degree	even		
	odd		

LT #3
Polynomials have unbounded end behavior.

$\lim_{x\to\infty} f(x)=\infty$ or $-\infty$

$\lim_{x\to-\infty} f(x)=\infty$ or $-\infty$

"The limit as x approaches ∞ of f(x) is..."

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Rational Functions and End Behavior (Activity: How Much Anesthesia Should the Patient Get?)

- Interpret the behavior of a rational function in context, specifically its horizontal asymptote.
- Determine the end behavior of a rational function by comparing the dominance of the polynomials in the numerator and denominator.
- Explain why the end behavior of a rational function is determined by the quotient of the leading terms in the numerator and denominator.

QuickNotes

LT#1: Rational functions have the form $h(x) = \frac{f(x)}{g(x)}$ where f and g are polynomials and can be used to model real-world relationships.

LT#2: Comparing the degrees of f and g indicates whether the numerator or denominator is dominating.

LT#3: To determine end behavior, can analyze a simpler function given by the ratio of the leading terms.

Ex: $\frac{3x^5+2x^2}{-2x^2}$ acts like $\frac{3x^5}{-2x^2} = \frac{-3}{2}x^3$ for large values of x.

$\frac{2x^4+5x}{x^4-1}$ acts like $\frac{2x^4}{x^4} = 2$ for large values of x H.A at $y = 2$

$\lim_{x\to\infty} h(x) = 2$

Graphs of Rational Functions (Activity: The "Hole" Truth)

- Identify key features of a rational function including its domain, intercepts, holes, and vertical asymptotes from its graph and equation in factored form.
- Use one-sided limit notation to describe the behavior of a rational function near a vertical asymptote.
- Determine the y-value of a hole by examining function outputs at input values sufficiently close to the x-value of the hole.

QuickNotes

Let $f(x) = \frac{(x-a)(x-c)}{(x-b)(x-c)}$ * Always start by factoring!

Then

$f(a) = \frac{0}{\text{non-zero}}$ and $x=a$ is a zero/x-intercept

$f(b) = \frac{\text{non-zero}}{0}$ and $x=b$ is a vertical asymptote.

$f(c) = \frac{0}{0}$ and there is a hole at $x=c$

To find y-value of hole, examine outputs very close to $x=c$.

Factored and Standard Forms of Polynomials (Activity: Changing Forms)

- Describe the advantages of writing a polynomial in factored form versus standard form.
- Convert polynomials from factored to standard forms and vice versa.
- Explain why when (x-k) is a factor of a polynomial, x=k is a zero of the polynomial
- Find all zeros of a polynomial function by hand or using technology.

QuickNotes

LT #1-3

Standard form	Factored form
• $y=ax^3+bx^2+cx+d$ • can easily see degree + y-intercept • must use graph to see x-intercepts	• $y=a(x-p)(x-q)(x-k)$ • can easily see zeros/x-intercepts are at $x=p$, $x=q$, and $x=k$ since substituting in these values make the whole expression $=0$

LT #4

Knowing a zero/x-intercept gives you a factor. Dividing by a known factor helps you find remaining factors, + identify remaining zeros.

Equivalent Representations of Rational Functions (Activity: Let's Be Rational!)

- Expand the ideas of factors, dividends, divisors, quotients, and remainders from numbers to functions.
- Divide polynomials using an area model.
- Explain why rewriting a rational function in equivalent ways can reveal different characteristics of the function, including slant asymptotes.

QuickNotes

Rational functions represent division problems.

dividend → $\dfrac{f(x)}{g(x)} = q(x) + \dfrac{r}{g(x)}$ ← remainder $\qquad f(x) = g(x)\cdot q(x) + r$

divisor →

quotient

As x increases or decreases without bound, $\dfrac{f(x)}{g(x)}$ is determined by the quotient, $q(x)$. If $q(x)$ is linear, the rational function has a slant asymptote of $y=q(x)$

To find the quotient, use an area model to divide f by g. Identify the remainder.

The Binomial Theorem (Activity: Where Are the Like Terms?)

- Generalize patterns for the expansion of binomials and explain the connection to the entries of Pascal's triangle.
- Expand binomial expressions using the Binomial Theorem.

QuickNotes

The terms in a binomial expansion follow a predictable pattern because of the distributive property.

$$(a+b)^n = \_a^nb^0 + \_\_a^{n-1}b^1 + \_\_a^{n-2}b^2 + \ldots + \_\_ab^{n-1} + \_b^n$$

The coefficients are determined by the sums of the previous expansion and can be seen in Pascal's triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

Note: a and b can be numbers or variables!

Calc Medic QuickNotes for Unit 3: Constructing Functions

A Library of Parent Functions (Activity: Building a Library)

- Understand that the parent function represents the most basic function in a family of functions.
- Describe the key features of six parent functions: identity, absolute value, square root, quadratic, cubic, and reciprocal.
- Analyze and compare the key features of parent functions.

QuickNotes

A parent function is the simplest function of a family of functions.

Famillies of functions (quadratic, linear, square root, etc.) share similar algebraic properties, graphs, and behaviors.

Transformations of Functions (Activity: What's My Transformation?)

- Construct a new function by applying translations, dilations, and reflections to a parent function.
- Given an equation or graph of a transformed function, describe the transformations that occurred from the parent function.
- Determine the domain and range of a transformed function.

translation

Transformation	Function Notation	Characteristics
Vertical Shift	$f(x)+c$	Moves up when… $c>0$ Moves down when… $c<0$
Horizontal Shift	$f(x+c)$	Moves right when… $c<0$ Moves left when… $c>0$
Vertical Dilations	$c\,f(x)$	Dilation factor: $\lvert c\rvert$ Stretches in the y-direction when… $\lvert c\rvert>1$ Shrinks/compresses in the y-direction when… $0<\lvert c\rvert<1$
Horizontal Dilations	$f(cx)$	Dilation factor: $\left\lvert \frac{1}{c}\right\rvert$ Shrinks/compresses in the x-direction when… $\lvert c\rvert>1$ Stretches in the x-direction when… $0<\lvert c\rvert<1$
Reflection over x-axis	$-f(x)$	$(x,y)\rightarrow(x,-y)$
Reflection over y-axis	$f(-x)$	$(x,y)\rightarrow(-x,y)$

Piecewise Functions (Activity: How Much do I Pay for 4G Data?)

- Interpret and evaluate functions that have different rules for certain intervals of the domain.
- Graph piecewise defined functions.
- Write equations for piecewise-defined functions given a graph or from a context.

QuickNotes

Piecewise defined functions have different rules for certain intervals of the domain.

To evaluate a piecewise function:
- determine what interval your input belongs to
- substitute input into that equation only

Selecting a Function Model (Activity: Can You DTR?)

- Identify an appropriate function type to construct a function model based on key observations about how the quantities in a scenario are changing.
- Describe the assumptions and restrictions related to a particular function model.

QuickNotes

LT # 1: Scenarios can be modeled with a particular function type based on how the two quantities vary with one another.

Common function models:
- linear
- polynomial

(look for constant 2^{nd}, 3^{rd}, 4^{th}... differences)
- rational (inversely proportional)

If there is more than one pattern, consider a piecewise function.

LT # 2: Functions modeling real-world data are always imperfect and oversimplify scenarios. Consider underlying assumptions being made + domain + range restrictions based on context.

Constructing a Function Model (Activity: Do Females Live Longer than Males?)

- Construct a function model based on the constraints of a mathematical or contextual scenario.
- Construct a function model using transformations from a parent function.
- Use rational functions to model quantities that are inversely proportional.
- Apply a function model to answer questions about a data set or contextual scenario.

QuickNotes

① Determine the function type.
② Solve for parameters of equation algebraically or with technology OR determine transformations from parent function to get equation.
③ State any domain restrictions.
④ Use model to make predictions.

Some functions can be determined from key characteristics (zeros, symmetry, behavior)

Some models are deterministic (only 1 right answer).

Calc Medic QuickNotes for Unit 4: Exponential Functions

Change in Arithmetic Sequences (Activity: #Goals)

- Understand that sequences are a special type of function whose domain is the positive integers.
- Write an explicit rule for arithmetic sequences using the common difference and any term in the sequence.
- Apply understanding of how arithmetic sequences grow to determine the common difference, find missing terms and reason about arithmetic sums.

QuickNotes

A sequence is a function whose inputs are the positive integers (1, 2, 3, 4 ...).

Arithmetic sequences have a common difference, d.

Explicit rule for nth term:

$$a_n = a_1 + d(n-1) \quad \text{or} \quad a_n = a_0 + dn \quad \text{or} \quad a_n = a_k + d(n-k)$$

nth term of the sequence

1st term

any term in the sequence

how far away from given term

Change in Geometric Sequences (Activity: Little Red's Crumby Day)

- Write an explicit rule for geometric sequences using the common ratio and any term in the sequence.
- Apply understanding of how geometric sequences grow and knowledge of exponents and roots to determine the common ratio and find missing terms.
- Compare arithmetic and geometric sequences.

QuickNotes

Geometric sequences have a common ratio, r.

Use repeated multiplication (or division) to get from one term to another (exponents!).

If $r > 1$ terms are increasing

If $0 < r < 1$ terms are decreasing

Increasing geometric sequences grow by a larger amount in each successive step.

Explicit rule for nth term:

$$a_n = a_1 \cdot r^{n-1} \quad \text{or} \quad a_n = a_0 \cdot r^n \quad \text{or} \quad a_n = a_k \cdot r^{n-k}$$

Change in Linear and Exponential Functions (Activity: Geri's Greeting Cards)

- Create linear and exponential functions using constant rates of change and constant proportions.
- Interpret the parameters of a linear and exponential function in context and to describe their growth patterns.
- Describe similarities and differences between linear and exponential functions.

QuickNotes

Linear functions have a constant rate of change.

$f(x) = \underset{\substack{\uparrow \\ \text{initial} \\ \text{value}}}{b} + \underset{\substack{\uparrow \\ \text{slope/} \\ \text{constant R.O.C}}}{mx}$ or $f(x) = y_1 + m(x - x_1)$

(x_1, y_1) is any input-output pair

* Think arithmetic

Exponential functions have a constant proportion.

$f(x) = \underset{\substack{\nearrow \\ \text{initial} \\ \text{value}}}{a} \cdot \underset{\substack{\nwarrow \\ \text{constant} \\ \text{proportion/} \\ \text{multiplier}}}{b^x}$ or $f(x) = y_1 \cdot b^{x-x_1}$

(x_1, y_1) is any input-output pair

* Think geometric

Linear + exponential functions can both be determined by only 2 input-output pairs.

Exponential Functions (Activity: Game, Set, Flat)

- Recognize scenarios that depict exponential growth or decay by identifying a fixed percent change or common ratio.
- Write equations of the form $y = ab^x$ to model scenarios that grow or decay by a fixed percent or factor.

QuickNotes

Scenarios depicting exponential growth or decay have a fixed percent change or common ratio.

$f(x) = \underset{\substack{\nearrow \\ \text{initial} \\ \text{value}}}{a} \underset{\substack{\nwarrow \\ \text{growth or} \\ \text{decay factor}}}{b^x}$

If $0 < b < 1 \Rightarrow$ exponential decay

If $b > 1 \Rightarrow$ exponential growth

If change is given as a percent, write percent as decimal rate, r.

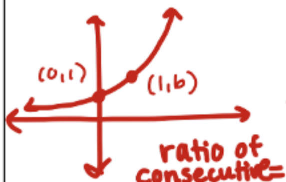
$\underset{\text{(growth)}}{b = 1 + r}$ or $\underset{\text{(decay)}}{b = 1 - r}$

Graphing and Manipulating Exponential Functions (Activity: Exponential Match-Up)

- Graph functions of the form $y = ab^x$ and identify key characteristics including end behavior, concavity, domain and range, asymptotes, and intercepts.
- Determine the growth factor of an exponential function from its graph, including when the function has been transformed.
- Apply knowledge of transformations to exponential functions.
- Explain using exponent properties and transformations why two exponential functions are equivalent.

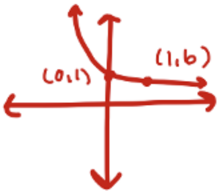
QuickNotes

Parent exponential functions have the form $p(x)=b^x$.



$b>1$ = growth

$\lim_{x\to\infty} p(x)=\infty$



$0<b<1$ = decay

$\lim_{x\to\infty} p(x)=0$

ratio of consecutive terms $= \frac{p(n)}{p(n-1)} = b$

Transformations reflect, shift, + dilate the graph.

$f(x) = a\cdot b^{x-h} + k$ $\quad \frac{f(n)}{f(n-1)} \neq b$

Exponent properties reveal that different transformations can have the same effect!

Modeling with the Natural Base, "e" (Activity: How Do You Grow Your Money?)

- Describe the effects of compounding interest quarterly, monthly, weekly, daily, and continually and make use of structure to arrive at the compound interest formula.
- Use an exponential model to make predictions about the dependent variable.
- Understand "e" as the base rate of growth for all continually growing processes.

QuickNotes

Compound Interest

$P(t) = P_0\left(1+\frac{r}{n}\right)^{nt}$

P_0 = initial amount invested

r = interest rate (as a decimal)

n = # of compoundings/year

t = # of years

Continuous compounding: $\lim_{n\to\infty} P(t) = P_0e^{rt}$

"e" is the base rate of growth for all continually growing processes

$e \approx 2.718$

Many contextual scenarios exhibit continuous exponential growth + are modeled w/ the natural base.

$y = ae^{kt}$

Constructing Exponential Models (Activity: How Often Should You Take DayQuil?)

- Construct exponential models from an initial value and ratio or from two input-output pairs.
- Use an exponential model to make predictions about the dependent variable.
- Understand how equivalent forms of an exponential function can reveal different properties about its growth rate.

QuickNotes

Exponential functions model growth patterns where successive output values over equal input intervals are proportional, using an initial value + ratio or 2 input-output pairs.

Equivalent forms can reveal different properties of the growth rate.

EX: $f(t) = 2^{t/7}$ describes a quantity that doubles every 7 years whereas $f(t) = (2^{1/7})^t = 1.104^t$ shows the growth factor per year (1.104) and the % change (10.4% each year)

Using Regression Models (Activity: Eating Out vs. Eating at Home)

- Use the characteristics of a data set to decide whether a linear, quadratic, or exponential model is most appropriate.
- Create a regression model for a scenario using technology.
- Use a residual plot to validate whether a given model was appropriate.

QuickNotes

A regression model is a function that describes the relationship between an independent + dependent variable given a 2 or more variable data set, usually found with technology.

The type of regression model (linear, quadratic, exponential) is chosen by the user.

least squares regression model

The model is considered appropriate if the graph of the residuals appears without pattern.

Residual value is the difference between the actual value + the value predicted by the model.

Calc Medic QuickNotes for Unit 5: Logarithmic Functions

Compositions of Functions (Activity: How Much Does It Cost to Tile a Pool?)

- Understand that when two functions are composed, the output of one function becomes the input of the second function.
- Write equations for compositions of functions.
- Decompose a complicated function into a composite of two or more functions.
- Reason about the domain of a composition of functions.

QuickNotes

LT#1 A composite function is made up of 2 or more functions where the output of one function becomes the input of the other.

$$\begin{tikzcd} x \arrow[r, bend left] & f(x) \arrow[r, bend left] & g(f(x)) \end{tikzcd}$$

LT#2
To write an equation for $g(f(x))$ substitute the entire expression for $f(x)$ for every instance of x in g.

LT#4 The domain of $g(f(x))$ is the subset of f's domain that will produce outputs in the domain of g.

LT#3: Complicated functions can be decomposed into simpler functions.

Ex: $(x+7)^2$ is a composition of $f(x) = x+7$ and $g(x) = x^2$

$g(f(x)) = (x+7)^2$

Intro to Inverse Functions (Activity: How Much Should You Feed Your Puppy?)

- Repeatedly solve equations of the form $f(x) = c$ to recognize the need for a function that "undoes" the original function, i.e. to find the value in the domain that generates a certain output.
- Understand the relationship between the inputs and outputs of a function and its inverse and use this to evaluate inverse functions.
- Find an inverse function algebraically.
- Verify by composition that one function is the inverse of another.

QuickNotes

Inverse functions "undo" the original function.

$f^{-1}(f(c) = c \qquad f(f^{-1}(c)) = c$

To find an inverse equation of $f(x)$

1) Replace $f(x)$ with y
2) Solve for the other variable. (dependent variable of f^{-1})
3) Rewrite w/ inverse notation.

Graphs of Inverse Functions (Activity: How Long to Reach the Summit?)

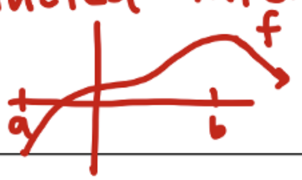
- Understand why a function must be one-to-one, or invertible, in order for the inverse mapping to be a function.
- Explore relationships between the graph of a function and its inverse, including their domains and ranges.

QuickNotes

In order for f to have an inverse that's also a function, f must be one-to-one (no repeated y-values)

This can be checked w/ a horizontal line test.

A function can be one-to-one on a restricted interval of the domain.



f is one-to-one on (a, b) but not for all $\mathbb{R}$

Domain of f is range of f^{-1}

Range of f is domain of f^{-1}

Inverses of Exponential Functions (Activity: The Mystery Function)

- Understand that a logarithm represents the exponent to which the base must be raised in order to attain the input value; use this understanding to evaluate logarithmic expressions.
- Use exponential and logarithmic forms to write equivalent statements about powers.
- Understand the inverse relationship between how inputs and outputs change in exponential versus logarithmic functions.
- Understand the inverse relationship between exponential and logarithmic functions of the same base, including the natural base, e.

QuickNotes

The output of a logarithmic function tells you what exponent you must raise the base to, to arrive at the input.

$\log_b x = y \iff b^y = x$

Logarithmic form   Exponential form

If no subscript is given, assume $b = 10$.

If the base is e, use the natural log: $\log_e x \Rightarrow \ln x$

Domain: $x > 0$ (since powers of b are always positive)

Range: $\mathbb{R}$ (since exponents can be neg or pos)

Graphs of Logarithmic Functions (Activity: Lumberjack Graphs)

- Describe key features (domain, range, asymptotes, concavity, and end behavior) of the graph of a parent logarithmic function, $y = \log_b x$.
- Sketch parent logarithmic functions and their transformations.
- Connect key features on the graphs of exponential and logarithmic functions.

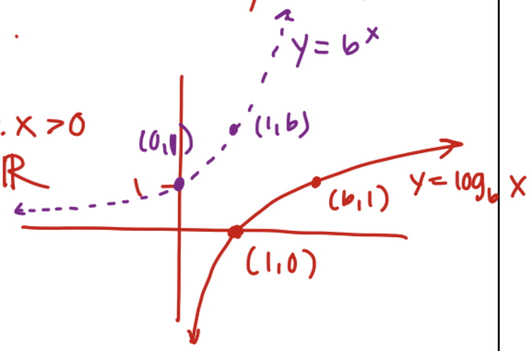
QuickNotes

Parent logarithmic functions look like $y=\log_b x$ for any base $b>0$, $b \neq 1$.

2 key pts:
$(1,0) \rightarrow$ x-intercept
$(b,1) \rightarrow$ Identity

Domain: $x>0$
Range: $\mathbb{R}$

vertical asymptote: $x=0$



b^x and $\log_b x$ are inverses!
(This includes e^x and $\log_e x = \ln x$)

Logarithm Properties (Activity: Puzzling with Properties)

- Discover the sum, difference, and power properties of logarithms and use them to rewrite logarithmic expressions.
- Explain using logarithm properties and transformations why two logarithmic functions are equivalent.

QuickNotes

LT#1:

CONDENSE

$$\log_b x + \log_b y = \log_b (xy)$$
$$\log_b x - \log_b y = \log_b \left(\frac{x}{y}\right)$$
$$a \log_b x = \log_b x^a$$

EXPAND

LT # 2

- Logarithm properties reveal that different transformations can have the same effect.
- Logarithms of different bases can be related to one another.

Solving Exponential and Logarithmic Equations (Activity: Looking for "?")

- Solve exponential and logarithmic equations by rewriting them in equivalent forms using properties.
- Identify and exclude extraneous solutions to exponential and logarithmic equations.
- Write equations for inverse functions by applying inverse operations.

QuickNotes

LT #1: Strategies for solving exponential + logarithmic equations.

- Rewrite in alternate form
- Isolate the variable using inverse operations
- Apply log properties to simplify expression
- Use one-to-one property
  - If $b^x = b^y$ then $x = y$
  - If $\log_b x = \log_b y$ then $x = y$

same output implies same input!

- Calculator: Find intersection of 2 graphs

LT # 2: Always check to make sure your solutions produce defined outputs in the original equation. (Are they in the domain?)

LT# 3: Finding an inverse function requires the same process since the independent variable is being isolated.

Modeling with Logarithmic Functions (Activity: What is a 6-Figure Salary?)

- Understand that a logarithmic model takes quantities that grow proportionally and assigns them output values that grow linearly.
- Identify situations that could be modeled with a logarithmic function.
- Construct logarithmic models using input-output pairs or transformations.
- Use logarithmic function models to predict values of the dependent variable.

QuickNotes

Logarithmic functions can be used to model situations where inputs grow multiplicatively and outputs grow additively.

Can construct a logarithmic model from:

- an x-intercept and proportion
- two input-output pairs
- verbal description of a scenario
- identifying transformations from parent function $P(x) = \log_b x$

Logarithmic models are often used to determine the independent variable of an exponential relationship

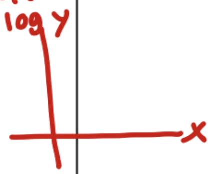
Semi-log Plots (Activity: How has GDP Per Capita Changed Over Time?)

- Understand that quantities exhibiting exponential growth or decay can be linearized using a log transformation.
- Interpret the parameters of exponential regression models and their associated linear regression models after a log transformation.

QuickNotes

If a function f shows exponential growth or decay, plotting (x, log (f(x))) will linearize the data.

The result is called a semi-log plot.



Interpreting parameters of regression models:

$\hat{y} = a \cdot b^x$

a = predicted y-intercept (in context)
b = predicted growth/decay factor

$\hat{y} = \log a + x \log b$

log a = predicted log of the y-intercept
lob b = predicted increase/decrease in log (dependent variable) per unit of independent variable.

* Must use "estimated" or "predicted"
* Must use context

Calc Medic QuickNotes for Unit 6: Exploring Sine and Cosine Functions

Periodic Phenomena (Activity: How Much Air is in Your Lungs?)

- Identify when two variables share a periodic relationship and construct their graph.
- Describe the key features of a periodic function based on a verbal description or graph.

QuickNotes

Two variables have a periodic relationship if the output values demonstrate a repeating pattern over equal length input intervals.

The graph of a periodic relationship is determined by a single cycle. The length of the input interval for one cycle is called the period, k.

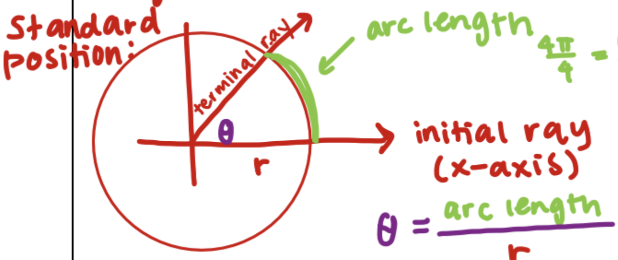
$$f(x+k) = f(x) \text{ for all } k.$$

Angles on the Coordinate Plane (Activity: Can You Measure That in Twizzlers?)

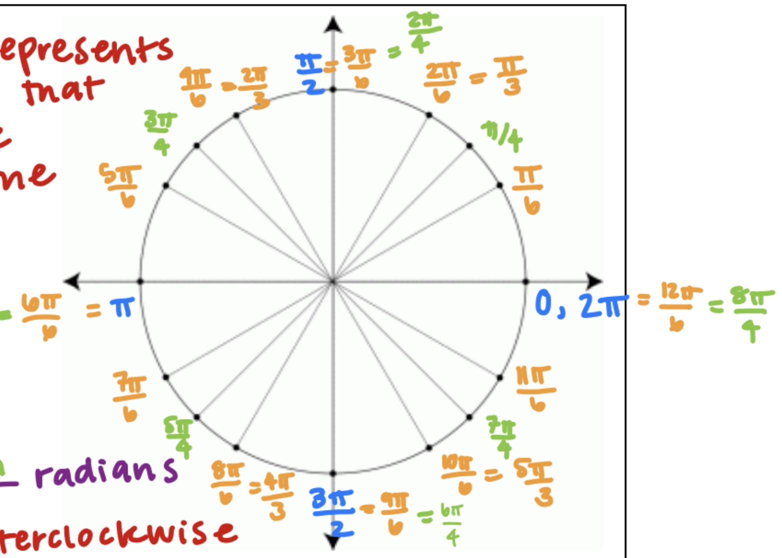
- Understand how to measure angles in standard position on the coordinate plane and their properties.
- Understand that a radian is an angle measure with an arc length of one radius.
- Label the angles on the unit circle in radians using proportional reasoning (i.e. partitions of semicircles).

QuickNotes

An angle in radians represents the # of radii lengths that fit on the arc of the circle intercepted by the angle.



$$\theta = \frac{\text{arc length}}{r} \text{ radians}$$



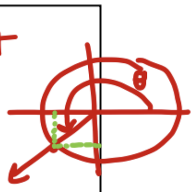
Angles are measured counterclockwise from the positive x-axis. Negative angles are measured clockwise.

Defining Sine, Cosine, and Tangent for Any Angle (Activity: Trig Ratios in the Wild)

- Extend the definition of sine, cosine, and tangent ratios to angles greater than 90˚ using the coordinate plane and horizontal and vertical displacement.
- Understand why in a unit circle, the sine and cosine ratios correspond with the y-value and x-value, respectively, of the point where the terminal ray intersects the circle.
- Understand that in a unit circle, the tangent of an angle is the ratio of the y-coordinate to the x-coordinate of the point where the terminal ray intersects the circle. Alternately, the tangent ratio is the slope of the terminal ray.
- Use symmetry to identify relationships between the sine, cosine, and tangent values of angles in all four quadrants.

QuickNotes

For angles greater than 90°, sine, cosine, + tangent are defined by displacement ratios from the point where the terminal ray of that angle intersects a circle.



$\sin\theta = \frac{\text{vertical displacement}}{\text{radius}}$ $\cos\theta = \frac{\text{horizontal displacement}}{\text{radius}}$ $\tan\theta = \frac{\text{vert.disp}}{\text{horiz. disp.}}$

On the unit circle where $r=1$, $\sin\theta = y$, $\cos\theta = x$, $\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$

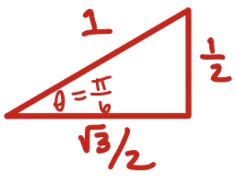
The sign of the sine, cosine, + tangent value is determined by the quadrant.

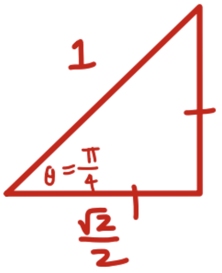
Coordinates on the Unit Circle (Activity: Coming Full Circle)

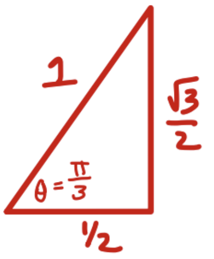
- Use special right triangles to determine the coordinates at key points on the unit circle.
- Evaluate sine, cosine, and tangent for key angles on the unit circle.
- Find coordinates of points on circles where r≠1.

QuickNotes

Use reflections of the special right triangles to determine ordered pairs at key pts on the unit circle.







To find coordinates of points on circles of radius r

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$ $\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

Graphs of Sine and Cosine (Activity: Spaghetti Waves)

- Construct graphs of the sine and cosine functions using values from the unit circle.
- Identify key characteristics for the parent functions y=sin x and y=cos x including domain, amplitude, midline, period, and symmetry.

QuickNotes

Two new parent functions

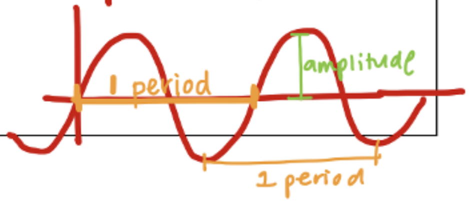
$y = \sin x$ — sine ratio: $\sin$; angle: x

Domain: $\mathbb{R}$
Range: $[-1,1]$
Period: 2π
Amplitude: 1

$y = \cos x$ — cosine ratio: $\cos$; angle: x

Domain: $\mathbb{R}$
Range: $[-1,1]$
Period: 2π
Amplitude: 1

- Amplitude is the distance between the midline and the maximum.
- Period is the length of one complete cycle



Transformations of Sine and Cosine (Activity: Which One Doesn't Belong?)

- Determine how the amplitude, period, domain, range, and midline of sinusoidal functions are affected by transformations.
- Graph transformed sine and cosine functions given an equation.

QuickNotes

$y = a\sin(b(x-c)) + d$ or $y = a\cos(b(x-c)) + d$

These are called sinusoidal functions.

a = vertical stretch, amplitude = $|a|$

b = horizontal shrink if $b > 1$
horizontal stretch if $0 < b < 1$

Period $= \dfrac{2\pi}{b}$ — 2π: period of parent function; b: horiz. dilation factor

c = horizontal shift / phase shift

d = vertical shift, affects midline + range!

Modeling with Trigonometric Functions (Activity: It’s Getting Hot Out Here!)

- Interpret a sinusoidal function's period, amplitude, midline, and range in context.
- Construct a trigonometric model based on data points and key features.

QuickNotes

- Real-world periodic behavior that repeats in a fixed time frame can be modeled with a sinusoidal function, $f(x) = \cos(b(x-c))+d$ or $f(x) = a\sin(b(x-c))+d$.
- The period represents the time it takes to complete one cycle. Period $= \frac{2\pi}{b}$
- The midline represents the average value of the function and vertical shift
- average + amplitude = max value $= d + |a|$
- average − amplitude = min value $= d - |a|$

Calc Medic QuickNotes for Unit 7: Working with Trigonometric Functions

The Tangent Function (Activity: How Are the Slopes Changing?)

- Understand how the tangent of an angle is determined by the slope of the terminal ray of the angle and use this to understand the behavior of the tangent function.
- Describe the key features of the graph of the tangent function, including its domain, range, x-intercepts, and period.
- Identify how the graph of the parent tangent function is affected by transformations.

QuickNotes

The tangent function represents the slope of the terminal ray of an angle, θ.

$f(\theta) = \tan\theta = \frac{\sin\theta}{\cos\theta}$

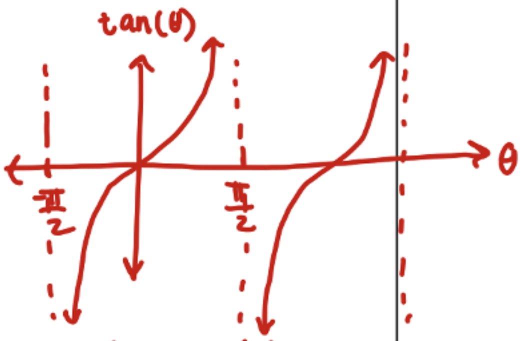
$\tan\theta = 0$ when $\sin\theta = 0$

$\tan\theta$ is not defined when $\cos\theta = 0$

Domain: $\mathbb{R}$, $x \neq \frac{\pi}{2} + \pi k$ where k is an integer

Range: $\mathbb{R}$

Period: π



Tangent functions can be transformed: $y = a\tan(b(\theta - c)) + d$

Period: $\frac{\pi}{b}$

Inverse Trig Functions (Activity: Caution: Restricted Area)

- Understand that inverse trigonometric functions input ratios and output angles. The input and output values are switched from their corresponding trigonometric functions.
- Explain why and how the domains of sine, cosine, and tangent must be restricted to create an inverse function.
- Evaluate inverse trig expressions.

QuickNotes

Inverse trig functions input ratios + output angles.

Since trig functions are periodic (many repeated outputs) we must restrict their domain to an interval on which they are one-to-one in order to define their inverse functions.

$f(x) = \sin^{-1}(x)$ or $\arcsin(x)$	$f(x) = \cos^{-1}(x)$ or $\arccos(x)$	$f(x) = \tan^{-1}(x)$ or $\arctan(x)$
D: $[-1, 1]$	D: $[-1, 1]$	D: $(-\infty, \infty)$
R: $[-\frac{\pi}{2}, \frac{\pi}{2}]$	R: $[0, \pi]$	R: $(-\frac{\pi}{2}, \frac{\pi}{2})$

Trigonometric Equations and Inequalities (Activity: It's Getting Hot Out Here! (Part 2))

- Extend the process of inverse operations to trigonometric equations and inequalities.
- Understand that using the unit circle will give infinite solutions to a trigonometric equation which may need to be restricted based on context and that an inverse trig function gives only one solution that may need to be expanded using symmetry.

QuickNotes

Inverse operations can be used to solve trigonometric equations and inequalities.

But the # of solutions needs to be carefully analyzed!

- The unit circle gives infinite solutions! These will need to be restricted based on the interval given or the context.
- Evaluating inverse trig functions on calculator gives only one solution. There may be more outside of the ranges of arcsin(x), arccos(x), and arctan(x). Find them using the symmetry of the unit circle.

The Secant, Cosecant, and Cotangent Functions (Activity: Is There More to Explore?)

- Define the secant, cosecant, and cotangent functions as the reciprocal of the cosine, sine, and tangent functions, respectively.
- Understand how the zeros, vertical asymptotes, and range are related for a trigonometric function and its reciprocal function.

QuickNotes

Secant, cosecant, and cotangent are the reciprocal trig functions.

$f(\theta) = \csc\theta = \frac{1}{\sin\theta}$	$f(\theta) = \sec\theta = \frac{1}{\cos\theta}$	$f(\theta) = \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$
Range: $(-\infty,-1]\cup[1,\infty)$	Range: $(-\infty,-1]\cup[1,\infty)$	Range: $(-\infty,\infty)$
Undefined when $\sin\theta = 0$	undefined when $\cos\theta = 0$	Undefined when $\sin\theta = 0$

The behavior of their graphs can be determined by understanding how reciprocal relationships work!

Trigonometric Relationships (Activity: Identity Crisis)

- Explore relationships between all six trigonometric functions, including the Pythagorean identities.
- Use identities to establish and verify other trigonometric relationships and solve trigonometric equations.

QuickNotes

Relationships between the six trig functions are called identities because they are true for any value of θ.

Equivalent forms of trig functions can reveal relationships, simplify complex expressions, help solve equations, + can be used to generate even more identities.

Pythagorean identities:

$\sin^2\theta + \cos^2\theta = 1$ $\quad 1 + \tan^2\theta = \sec^2\theta$ $\quad 1 + \cot^2\theta = \csc^2\theta$

Note: $\sin^2\theta = (\sin\theta)^2$

Angle Sum Identities (Activity: Break It Down!)

- Find exact values for the sine and cosine of angles not on the unit circle by writing the angle as a sum or difference of known angles.
- Use equivalent trigonometric expressions arising from the angle sum and double angle identities to solve equations.

QuickNotes

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

Why useful?

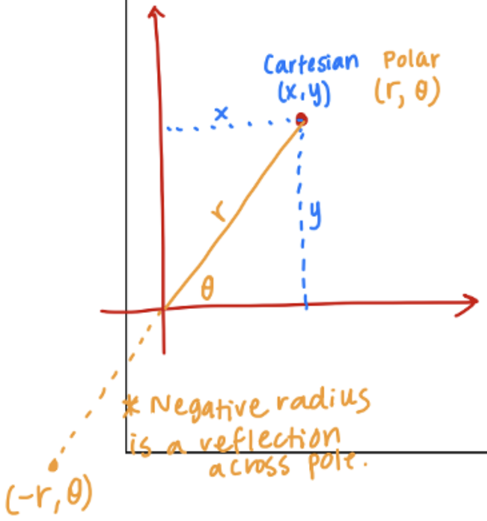
- Find the sine and cosine of unknown angles by decomposing the angle into two known angles.
- Equivalent trigonometric expressions can be used to solve a trig equation (just like log or exponent properties)

Calc Medic QuickNotes for Unit 8: Polar Functions

Polar Coordinates (Activity: Supervising the Sky)

- Understand that polar coordinates give an alternate method for locating points using a distance from the origin and an angle from the positive x-axis.
- Use coterminal angles and reflected radii to name polar points in multiple ways.
- Convert between polar and rectangular coordinates.

QuickNotes



*Negative radius is a reflection across pole.

Polar coordinates give locations w/ a distance from the pole (r) and a CCW rotation from x-axis/ polar axis (θ)

There are many polar coordinates for the same pt.

Conversions

C → P

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

calc gives $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, adjust for quadrant!

P → C

$\cos\theta = \frac{x}{r}$

$\sin\theta = \frac{y}{r}$

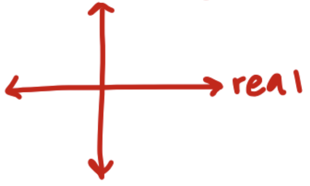
$$\boxed{\begin{gathered} x = r\cos\theta \\ y = r\sin\theta \end{gathered}}$$

Complex Numbers (Activity: Why So Complex?)

- Represent complex numbers on the complex plane in rectangular and polar form.
- Given a complex number in rectangular or polar form, identify its real and imaginary component.

QuickNotes

A complex number, z, has a real and imaginary component + can be graphed on the complex plane



If coordinates for the point are given in rectangular form $(\underset{x}{\underset{\uparrow}{a}},\underset{y}{\underset{\uparrow}{b}})$ then

$z = a + bi$

If coordinates are given in polar form (r,θ) then

$z = r\cos\theta + ir\sin\theta$ or $z = r(\cos\theta + i\sin\theta)$

Polar Graphs: Circles and Roses (Activity: A Polar Phenomenon (Part 1))

- Understand that polar functions input angle measures and output radii and point-by-point graphing can be used to construct their graphs.
- Identify the number, length, and location of petals of a polar rose from the values of the parameters, a and n.
- Describe key features of the polar graphs of circles and roses including their symmetry, domain, and range.

QuickNotes

Small circles

diameter

$r = \boxed{a}\cos\theta$

$a > 0$ right of y-axis

$a < 0$ left of y-axis

polar axis symmetry

diameter

$r = \boxed{a}\sin\theta$

$a > 0$ above polar axis

$a < 0$ below polar axis

y-axis/$\theta = \pi/2$ symmetry

Roses

$r = a\cos(n\theta)$

polar axis symmetry

1st petal at $\theta = 0$

$r = a\sin(n\theta)$

$\theta = \pi/2$ symmetry

1st petal at $\theta = \pi/2n$

If n is odd → n petals

If n is even → 2n petals

Polar Graphs: Limaçons (Activity: A Polar Phenomenon (Part 2))

- Identify special types of limaçons by comparing values of the parameters, a and b.
- Describe key features of the graphs of limaçons including symmetry, intercepts, domain, and range, and maximum and minimum values.

QuickNotes

Limacons

$r = a + b\cos\theta$ or $r = a + b\sin\theta$

$|a| < |b|$ loop

$|a| = |b|$ cardioid

$|a| > |b|$ dent/clam

Max radius: $|a| + |b|$

Min radius: $|a| - |b|$

Rates of Change of Polar Functions (Activity: What's Going On in This Graph?)

- Analyze and interpret key features of polar functions including intervals of increasing/decreasing and extrema.
- Find and interpret the average rate of change of a polar function.

QuickNotes polar functions can be analyzed like any other function!

- Extrema → max and min
- Intervals of increasing/decreasing
- Distance from the origin

	r is pos	r is neg
r is increasing	dist. from (0,0) is increasing	dist. from (0,0) is decreasing
r is decreasing	dist. from (0,0) is decreasing	dist. from (0,0) is increasing

The average rate of change of a polar function indicates the rate at which the radius is changing per radian.

$$\frac{\Delta r}{\Delta \theta} = \frac{f(\theta_2) - f(\theta_1)}{\theta_2 - \theta_1}$$