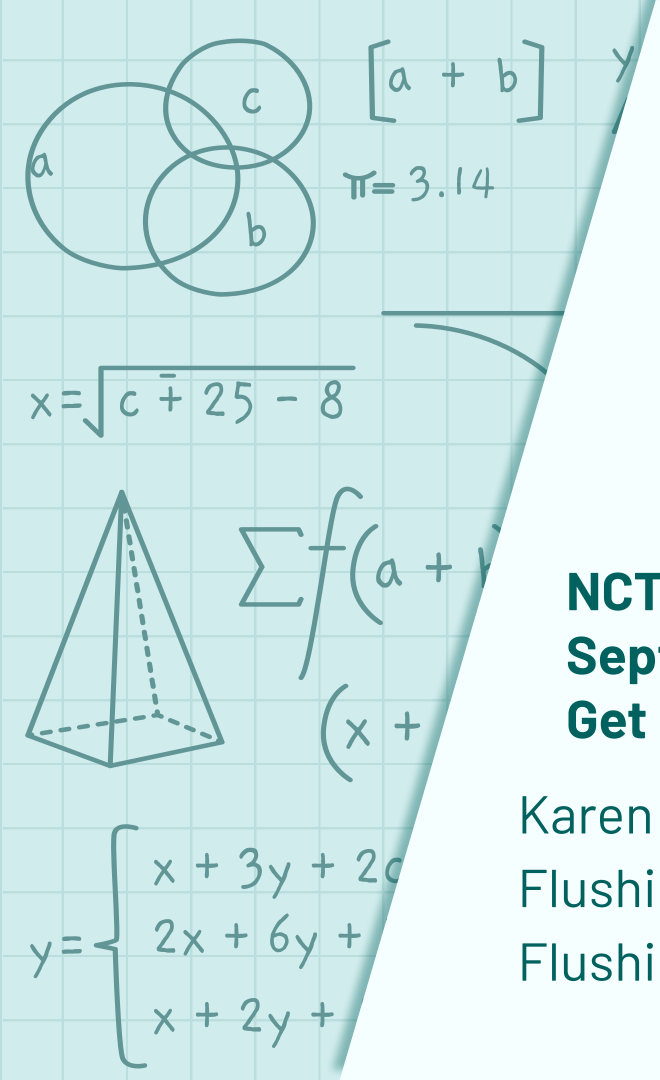

# Changing How We Teach Rates of Change

**NCTM Annual Conference**
**September 25, 2024**
**Get materials at bit.ly/NCTMrateofchange**

Karen Sleno
Flushing High School
Flushing, MI

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Math Medic

# We'd like to introduce ourselves!

## Karen

31-year teacher at Flushing High School in Michigan

AP Calculus Consultant (since 2015)

AP Reading since 2006 (Exam Leader since 2022)

Lucky enough to work with Sarah since 2022!

## Sarah

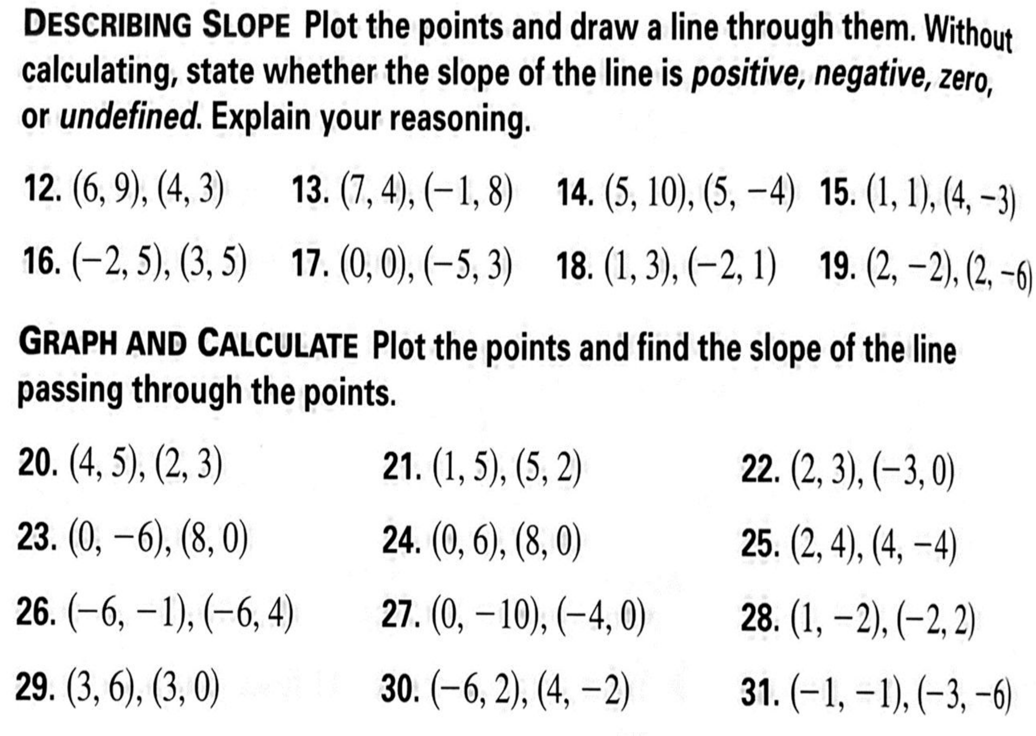
Taught at East Kentwood High School in Michigan

Content creator and lesson designer for Math Medic

AP Daily presenter for AP Calc and AP Precalculus

Met Karen at an APSI as a first year Calc teacher

# What is slope?



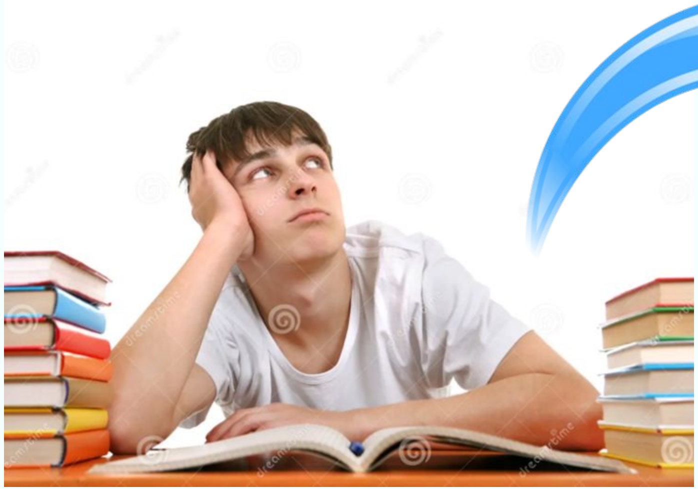
**DESCRIBING SLOPE** Plot the points and draw a line through them. Without calculating, state whether the slope of the line is *positive, negative, zero,* or *undefined.* Explain your reasoning.

**12.** (6, 9), (4, 3)
**13.** (7, 4), (−1, 8)
**14.** (5, 10), (5, −4)
**15.** (1, 1), (4, −3)
**16.** (−2, 5), (3, 5)
**17.** (0, 0), (−5, 3)
**18.** (1, 3), (−2, 1)
**19.** (2, −2), (2, −6)

**GRAPH AND CALCULATE** Plot the points and find the slope of the line passing through the points.

**20.** (4, 5), (2, 3)
**21.** (1, 5), (5, 2)
**22.** (2, 3), (−3, 0)
**23.** (0, −6), (8, 0)
**24.** (0, 6), (8, 0)
**25.** (2, 4), (4, −4)
**26.** (−6, −1), (−6, 4)
**27.** (0, −10), (−4, 0)
**28.** (1, −2), (−2, 2)
**29.** (3, 6), (3, 0)
**30.** (−6, 2), (4, −2)
**31.** (−1, −1), (−3, −6)

# Time for a change!



# What are rates of change?

# Support for Big Ideas vs. Checklist of Standards

“The beauty of teaching and learning a smaller set of connected, bigger ideas is that teachers and parents have more time to go into depth on each idea, and they allow students to think conceptually.

When students can dive deeply into the mathematical concepts, they learn the same mathematics, but instead of learning disconnected methods piece by piece, they learn a set of connected ideas and methods through rich tasks.”

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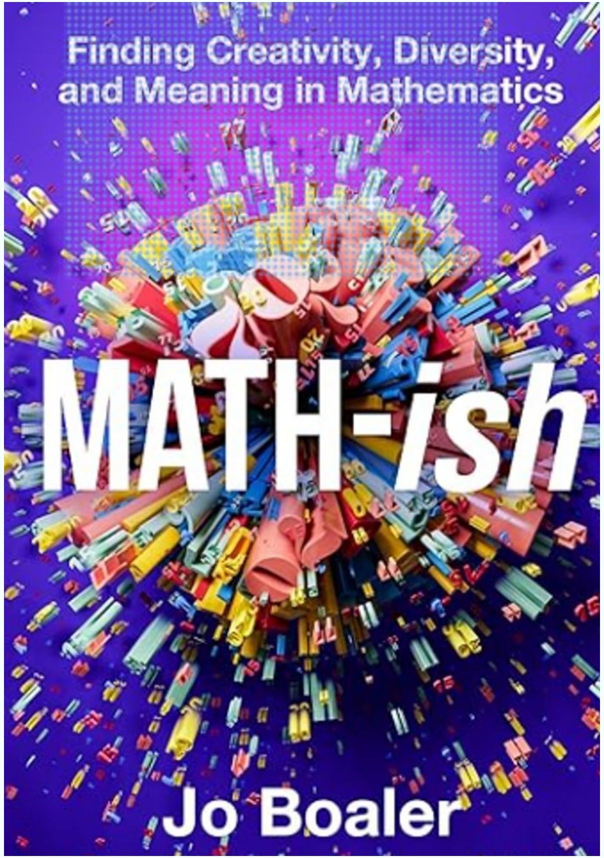
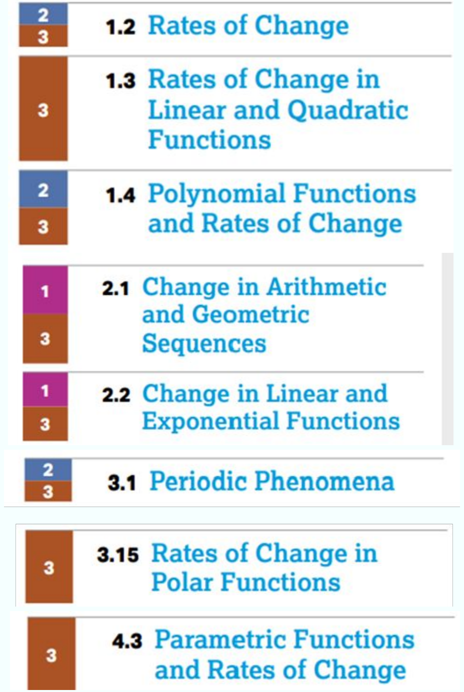


Published May 2024

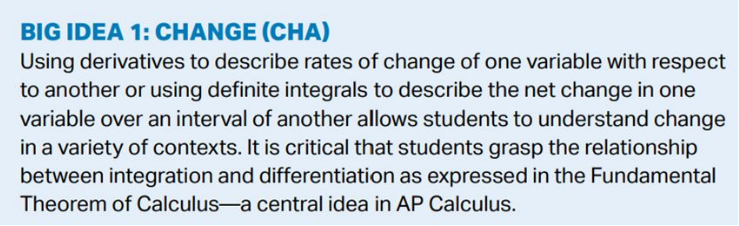
# Rates of Change = A Big Idea in AP Already!

## AP Precalculus

## AP Calculus



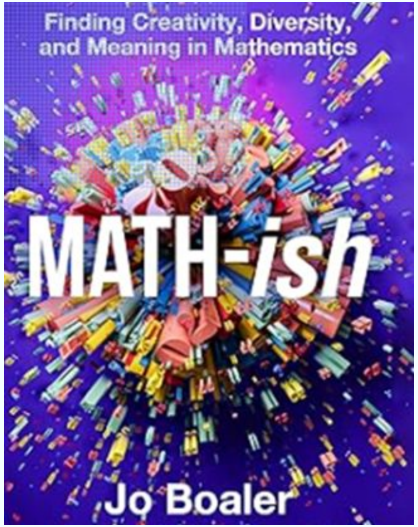
| | | |
|---|---|---|
| 2<br>3 | 1.2 | Rates of Change |
| 3 | 1.3 | Rates of Change in Linear and Quadratic Functions |
| 2<br>3 | 1.4 | Polynomial Functions and Rates of Change |
| 1<br>3 | 2.1 | Change in Arithmetic and Geometric Sequences |
| 1<br>3 | 2.2 | Change in Linear and Exponential Functions |
| 2<br>3 | 3.1 | Periodic Phenomena |
| 3 | 3.15 | Rates of Change in Polar Functions |
| 3 | 4.3 | Parametric Functions and Rates of Change |



**BIG IDEA 1: CHANGE (CHA)**

Using derivatives to describe rates of change of one variable with respect to another or using definite integrals to describe the net change in one variable over an interval of another allows students to understand change in a variety of contexts. It is critical that students grasp the relationship between integration and differentiation as expressed in the Fundamental Theorem of Calculus—a central idea in AP Calculus.

# Who learns better?



## Scenario #1

Teacher presents material and models a process

Students mimic the process with the teacher's support.

Students practice the process on their own.

## Scenario #2

Students are provided with a thinking activity to work on in groups but are not given mathematical directions on how to solve the problem.

After students work on the activity, the teacher formalizes the mathematical concepts.

Students apply the concepts to the activity.

# EFFL Lessons...

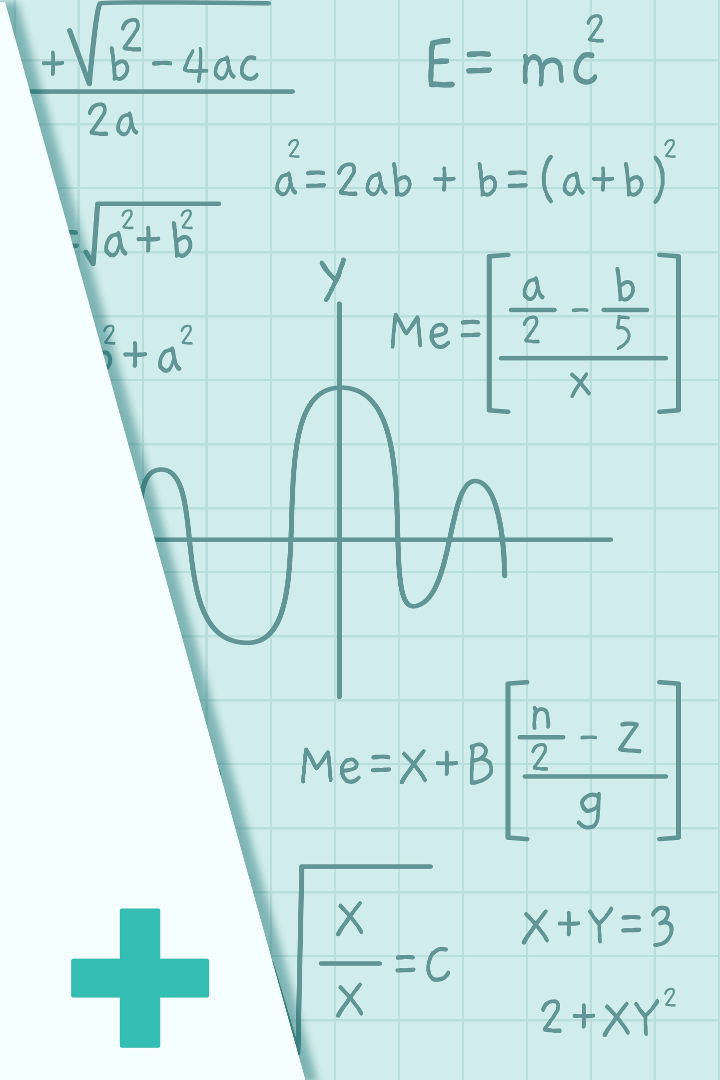
## courtesy of Math Medic

E xperience

F irst

F ormalize

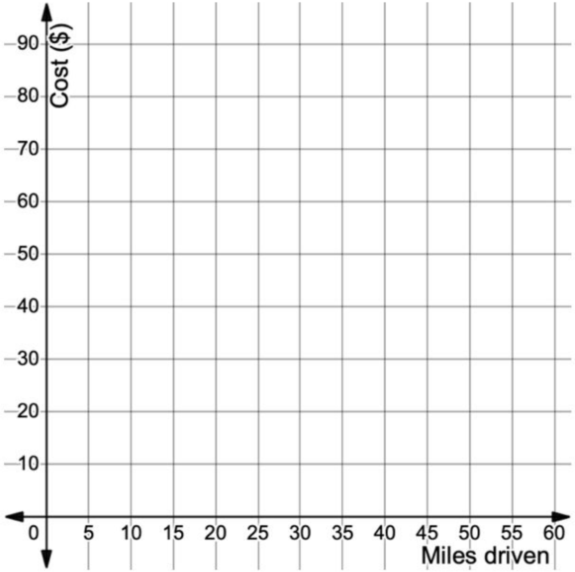
L ater



# Activity #1: Rates of Change in Linear Scenarios

**Math Medic**
**AP Precalculus Unit 1 Lesson 5**
**CED Topic 1.3**

2. Graph the points. What does the graph reveal about the relationship between the number of miles driven and the cost of the rental?



3. Is the rate of change of the cost increasing, decreasing, or staying constant? Explain.

4. U-Haul charges all its customers a minimum base price regardless of how many miles they drive. What is this base price? How do you know?

5. Write an equation for $C(m)$, the cost of renting the U-Haul and driving it for $m$ miles.

# The Takeaways...

AP

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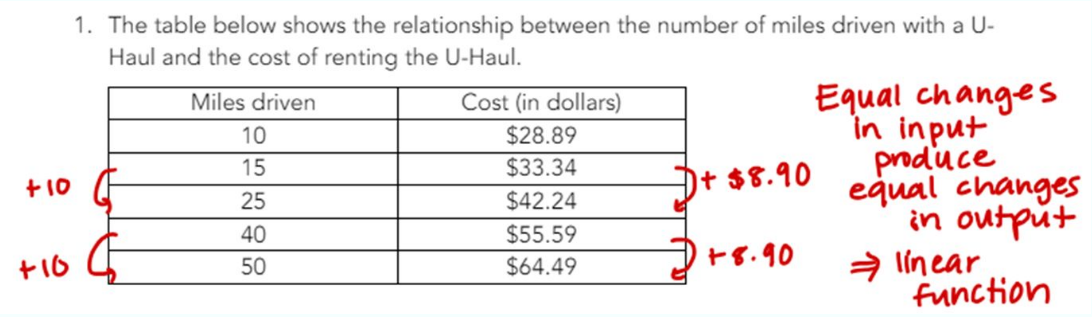


1. The table below shows the relationship between the number of miles driven with a U-Haul and the cost of renting the U-Haul.

| Miles driven | Cost (in dollars) |
|---|---|
| 10 | \$28.89 |
| 15 | \$33.34 |
| 25 | \$42.24 |
| 40 | \$55.59 |
| 50 | \$64.49 |

+10

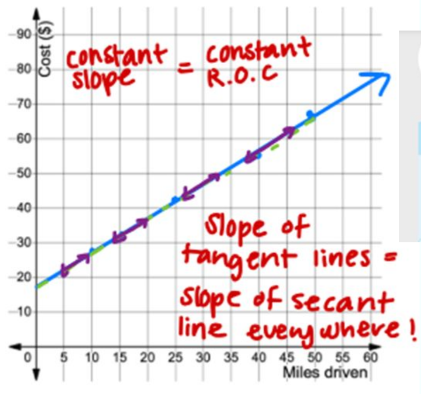
+10

+ \$8.90

+8.90

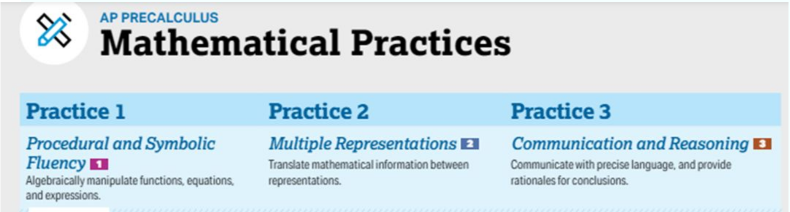
Equal changes in input produce equal changes in output

⇒ linear function



constant slope = constant R.O.C

Slope of tangent lines = slope of secant line everywhere!



AP PRECALCULUS

**Mathematical Practices**

| Practice 1 | Practice 2 | Practice 3 |
|---|---|---|
| *Procedural and Symbolic Fluency* 1 | *Multiple Representations* 2 | *Communication and Reasoning* 3 |
| Algebraically manipulate functions, equations, and expressions. | Translate mathematical information between representations. | Communicate with precise language, and provide rationales for conclusions. |

✓ ✓ ✓

# Activity #2: Rates of Change in Quadratic Scenarios

**Math Medic**
**AP Precalculus Unit 1 Lesson 6**
**CED Topic 1.3**

# First, time to…

# The Rules of EFFL

1. **NO Quitters**
2. **NO Spoilers**
3. **NO Loners**

# How Fast Does a Penny Fall from the Empire State Building?

A penny is dropped from the top of the Empire State Building, from a height of 1,250 feet. The height of the penny, in feet, $t$ seconds after it is dropped is given by the function $H(t) = 1250 - 16t^2$.

1. Complete the table to shov

+2 +2 +2 +2

| $t$ (seconds) | 0 | 2 | 4 | 6 | 8 |
|---|---|---|---|---|---|
| $H(t)$ (feet) | 1250 | 1186 | 994 | 674 | 226 |

−64 −192 −320 −448

Not constant

2. Find the exact time $t$ when the penny reaches the ground.

$H(t) = 0$ $1250 - 16t^2 = 0 \Rightarrow 1250 = 16t$

$78.125 = t^2$

$t = \sqrt{78.125} \approx 8.84$ second

3. Find the average rate of change in the penny's height during the total length of its drop.

$$\frac{-1250 \text{ ft}}{\sqrt{78.125} \text{ s}} \approx -141.42 \text{ ft/s}$$

Determine how many feet the penny fell during each two second interval. What do you notice?

The distance fallen over each 2-second interval grows by 128 ft.

The change in height follows a linear pattern.

64 ft, 192 ft, 320 ft, 448 ft

The number of feet fallen is increasing, not constant.

5. Complete the table with the average rate of change of the height of the penny.

| Time interval | $0 \le t \le 2$ | $2 \le t \le 4$ | $4 \le t \le 6$ | $6 \le t \le 8$ |
|---|---|---|---|---|
| Average rate of change of $H$ over that interval | -32 ft/s | -96 ft/s | -160 ft/s | -224 ft/s |

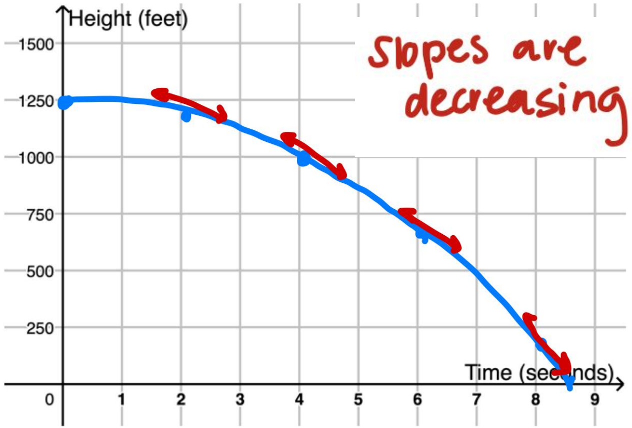
−64 ft/s    −64 ft/s    −64 ft/s

Constant change in the avg. ROC!

6. Is the penny speeding up, slowing down, or falling at a constant speed? How do you know?

The penny is speeding up. The average rates of change over each interval are getting more negative which implies a faster rate.

7. Graph $y = H(t)$.



Slopes are decreasing

8. Is the graph of $H$ concave up or concave down? What does this mean in the context of the problem?

Concave down. The rate that the penny's height changes is decreasing.
(slopes are getting more negative)

# Check Your Understanding

2. Selected values of a function $g$ are given. Complete the table so that $g$ is a quadratic function.

| $x$ | -8 | -4 | 0 | 4 | 8 | 12 | 16 | 20 |
|---|---|---|---|---|---|---|---|---|
| $g(x)$ | 1 | 14 | 24 | 31 | 35 | 36 | 34 | 29 |

+13 +10 +7 +4 +1 -2 -5

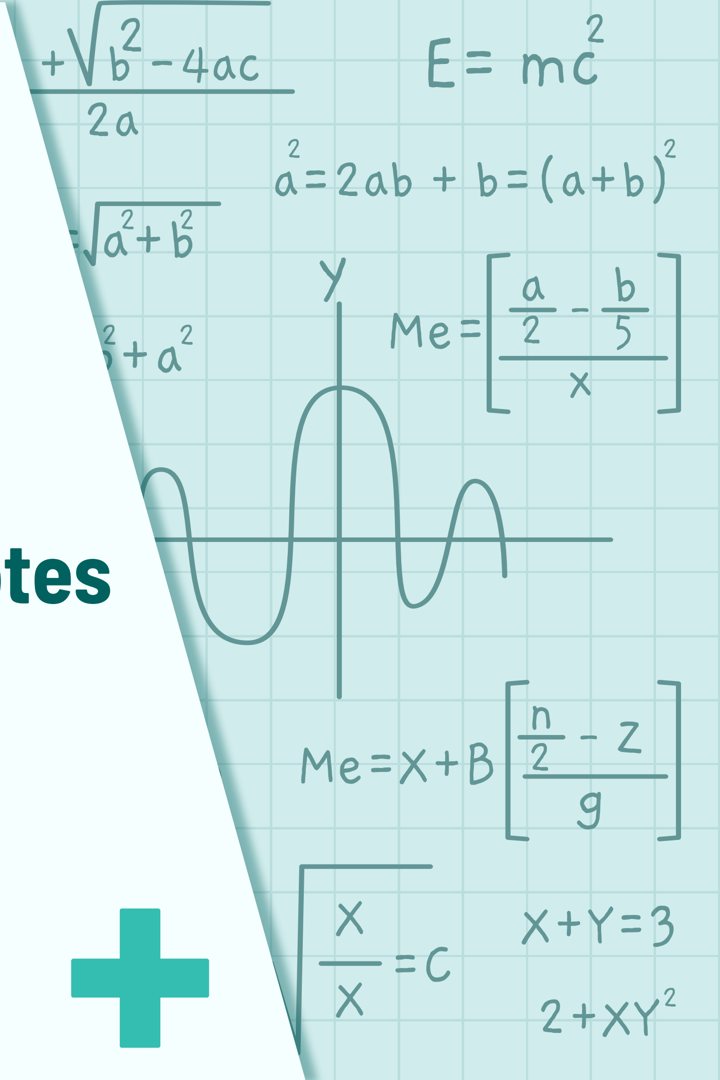
-3 -3 -3 -3 -3 -3

# WHAT HAVE WE LEARNED ABOUT QUADRATIC RELATIONSHIPS?

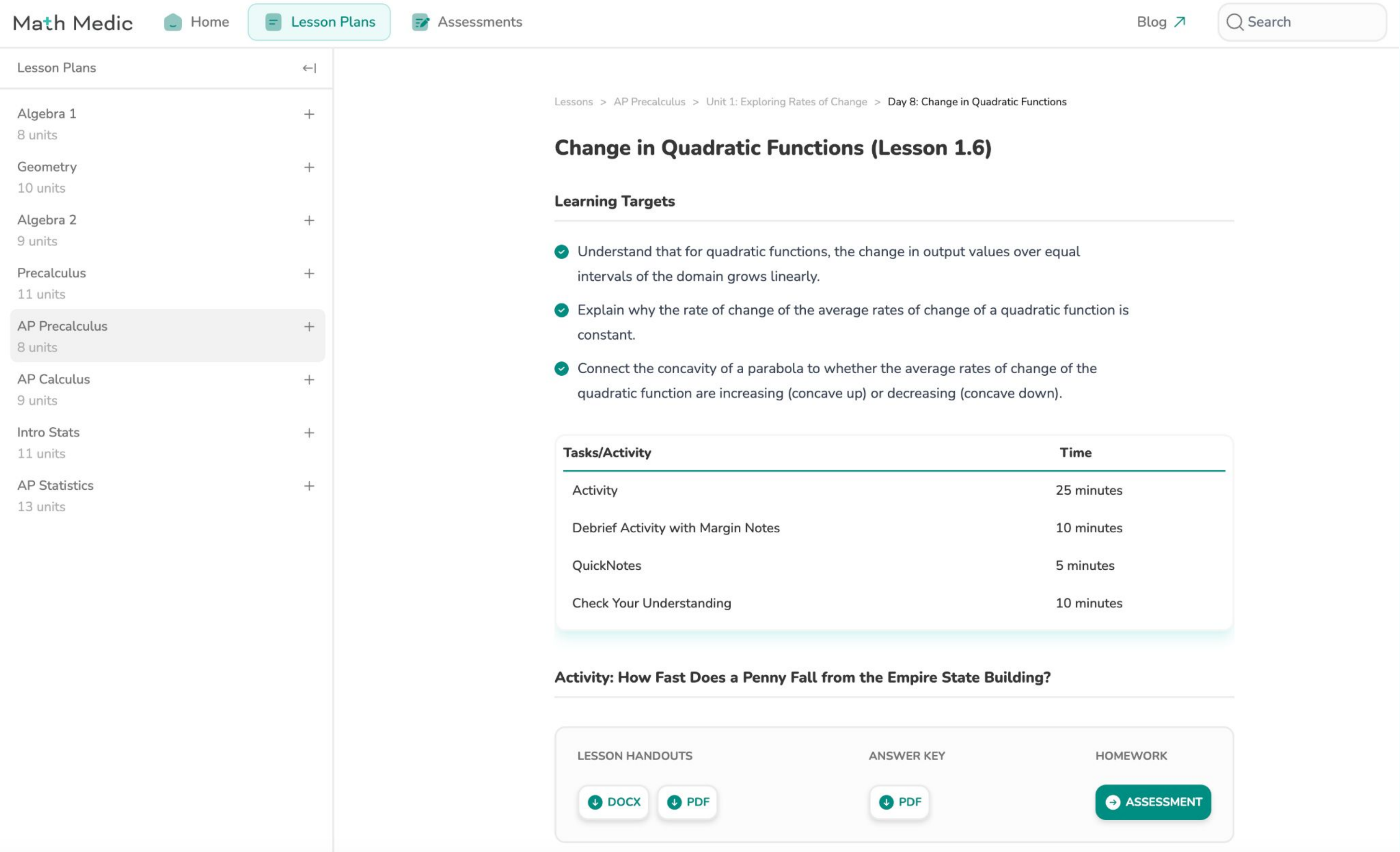
When the rates of change are **changing at a constant rate**, the relationship is quadratic!

# The 4 Parts of an EFFL Lesson

1. Activity
2. Debrief Activity: Margin Notes
3. Quick Notes
4. Check Your Understanding



# The Math Medic website...



## Change in Quadratic Functions (Lesson 1.6)

### Learning Targets

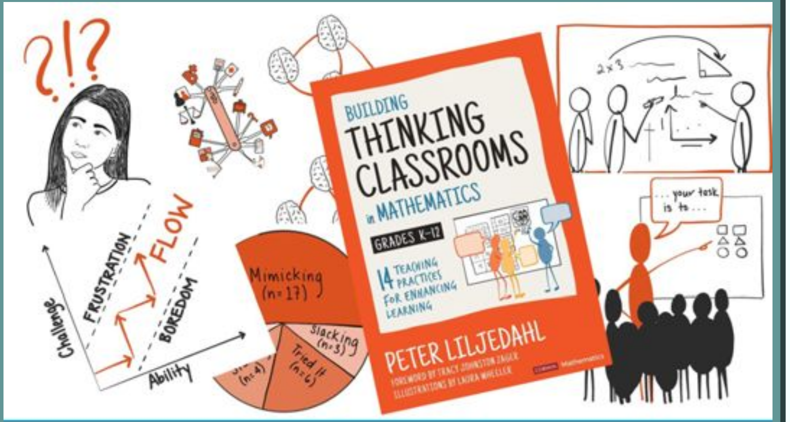
- Understand that for quadratic functions, the change in output values over equal intervals of the domain grows linearly.
- Explain why the rate of change of the average rates of change of a quadratic function is constant.
- Connect the concavity of a parabola to whether the average rates of change of the quadratic function are increasing (concave up) or decreasing (concave down).

| Tasks/Activity | Time |
|---|---|
| Activity | 25 minutes |
| Debrief Activity with Margin Notes | 10 minutes |
| QuickNotes | 5 minutes |
| Check Your Understanding | 10 minutes |

### Activity: How Fast Does a Penny Fall from the Empire State Building?

# Sounding Familiar?

# EFFL and BTC work well together!



- Highly engaging thinking tasks? ✓
- Visibly random groups? ✓
- Answer only "keep thinking questions"? ✓
- Consolidation vs. lecture? ✓
- Notes to your future forgetful self? ✓
- Check your understanding vs. traditional homework? ✓

# This ties in perfectly with research!

"Researchers have compared the approach used in most math classrooms—teaching methods, then students practice the methods in question—with a different approach...teachers give the students questions and tasks *before* they teach the methods they need to solve them.

All the studies show that this teaching approach brings about higher outcomes, and the researchers conclude that this happens because students get a greater opportunity to struggle—to think about and draw from the knowledge they have already developed."

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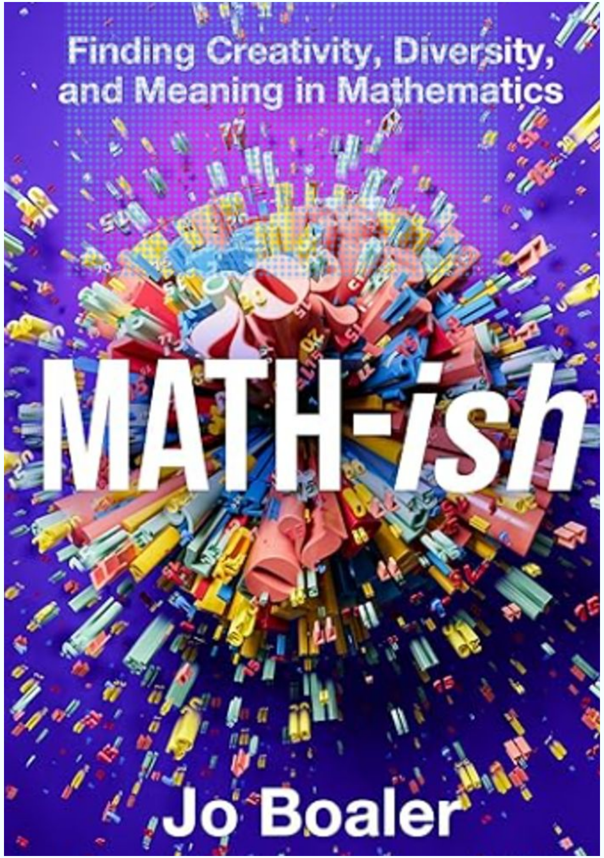
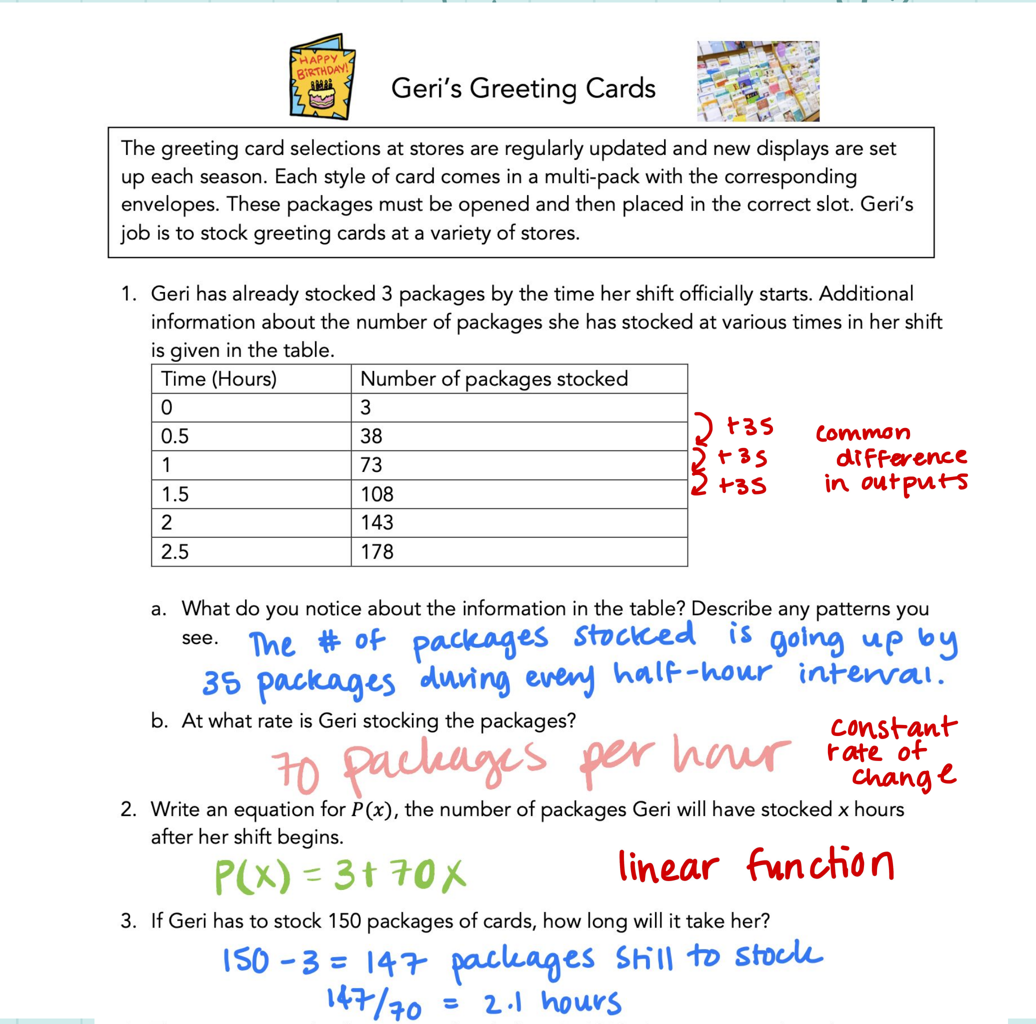


Published May 2024

# Activity #3: Rates of Change in Exponential Scenarios

**Math Medic**
**AP Precalculus Unit 4 Lesson 3**
**CED Topic 2.2**



# Geri's Greeting Cards

The greeting card selections at stores are regularly updated and new displays are set up each season. Each style of card comes in a multi-pack with the corresponding envelopes. These packages must be opened and then placed in the correct slot. Geri's job is to stock greeting cards at a variety of stores.

1. Geri has already stocked 3 packages by the time her shift officially starts. Additional information about the number of packages she has stocked at various times in her shift is given in the table.

| Time (Hours) | Number of packages stocked |
|---|---|
| 0 | 3 |
| 0.5 | 38 |
| 1 | 73 |
| 1.5 | 108 |
| 2 | 143 |
| 2.5 | 178 |

+35
+35
+35

common difference in outputs

   a. What do you notice about the information in the table? Describe any patterns you see. The # of packages stocked is going up by 35 packages during every half-hour interval.

   b. At what rate is Geri stocking the packages?

   70 packages per hour

   constant rate of change

2. Write an equation for $P(x)$, the number of packages Geri will have stocked x hours after her shift begins.

   $P(x) = 3 + 70x$

   linear function

3. If Geri has to stock 150 packages of cards, how long will it take her?

   $150 - 3 = 147$ packages still to stock

   $147/70 = 2.1$ hours

4. The greeting card industry is on the decline. In 2020, Americans purchased approximately 6.5 billion greeting cards each year. Between 2018 and 2023, this number has continued to decrease at approximately 2.8% per year. According to this estimate, how many greeting cards were purchased in 2021?

$6.5 - 0.028(6.5) = 6.318$ billion cards

5. What is the ratio of greeting card purchases from 2021 compared to 2020?

$$\frac{6.318}{6.5} = 0.972 = 1 - 0.028$$

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6. Complete the table.

| Year | Annual number of greeting cards purchased (in billions) |
|---|---|
| 2018 | 6.880 |
| 2019 | 6.687 |
| 2020 | 6.5 |
| 2021 | 6.318 |
| 2022 | 6.141 |
| 2023 | 5.969 |

$\div 0.972$
$\div 0.972$
$\times 0.972$
$\times 0.972$
$\times 0.972$

decrease by 2.8% means 97.2% remains from one year to the next

common ratio of 0.972

7. Is the number of greeting cards purchased annually decreasing by the same amount each year? Explain.

No, the year-to-year decrease is getting smaller. This is because 0.028 of a smaller amount gives a smaller decrease.

8. Write an equation for $G(x)$, the number of annual greeting cards purchased x years after 2018.

$$G(x) = 6.880(0.972)^{x}$$

or

$$G(x) = 6.5(\underline{0.972})^{x-2}$$

↑ constant proportion

Exponential Function

9. Compare your expressions for $P(x)$ and $G(x)$ from this activity. What makes them different? What similarities do they have?

They both follow a predictable pattern from one output to the next. The function P had a common difference and the function G had a common ratio. G had a constant % decrease whereas P had a constant amount increase.

Both functions were either only increasing or only decreasing.

P(x) looked like an arithmetic sequence whereas G(x) looked like a geometric sequence.

Both had an initial value.

P(x) outputs change additively

G(x) outputs change multiplicatively

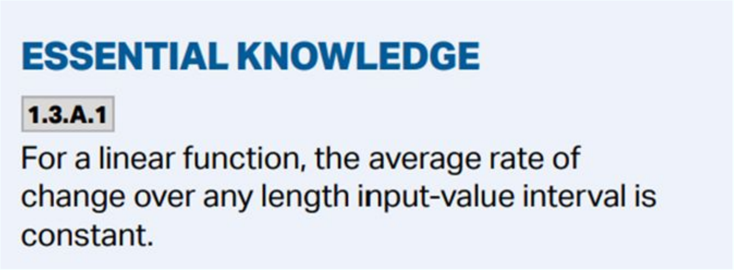
# What we now know....

TOPIC 1.3

## Rates of Change in Linear and Quadratic Functions

TOPIC 2.2

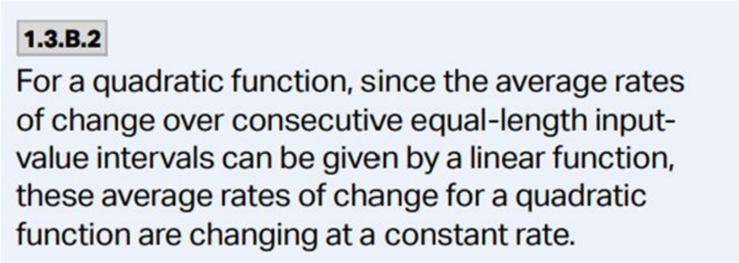
## Change in Linear and Exponential Functions



**ESSENTIAL KNOWLEDGE**

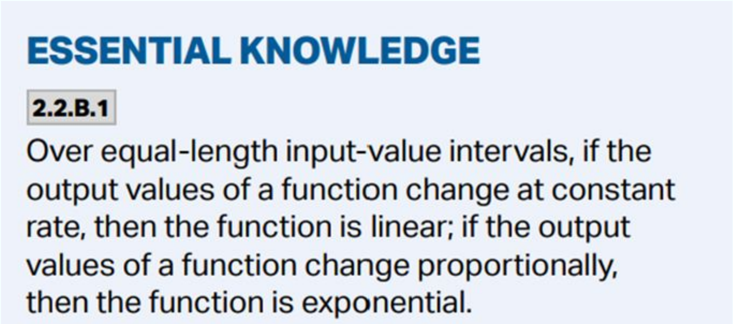
**1.3.A.1**

For a linear function, the average rate of change over any length input-value interval is constant.



**1.3.B.2**

For a quadratic function, since the average rates of change over consecutive equal-length input-value intervals can be given by a linear function, these average rates of change for a quadratic function are changing at a constant rate.



**ESSENTIAL KNOWLEDGE**

**2.2.B.1**

Over equal-length input-value intervals, if the output values of a function change at constant rate, then the function is linear; if the output values of a function change proportionally, then the function is exponential.

# Activity #4: Rates of Change in Calculus

**Math Medic**
**AP Calculus Unit 2 Lesson 2**
**CED Topic 2.2**

# SUPERSONIC FREEFALL

On October 14th, 2012, Austrian skydiver Felix Baumgartner broke a world record for a high-altitude dive when he ascended 127,850 feet in a helium balloon and then went into a free fall lasting more than 4 minutes.

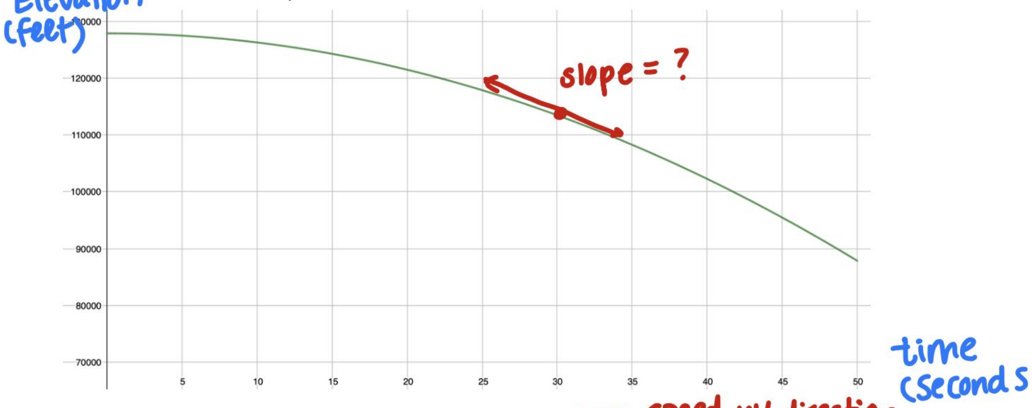
1. Baumgartner is in free fall for 4 minutes and 20 seconds (260 seconds) before he deploys his parachute at an elevation of 8,420 feet above sea level.
    a. What was the vertical distance of the freefall?

    127,850 − 8420 = 119,430 feet in freefall

    b. What was his average velocity during the freefall?

    $\frac{-119{,}430 \text{ ft}}{260 \text{ s}} = -459.346 \text{ ft/s}$

2. His elevation (in feet) above sea-level, $t$ seconds after stepping off the balloon can be approximated by $f(t) = 127850 - 16t^2$ for $0 \le t \le 50$.
    a. Look at the graph of $f(t)$ below. Label both axes.



    b. Was Baumgartner traveling at a constant velocity? How do you know?

    speed w/ direction

    No, there's not a constant slope.

    c. What time does it look like Baumgartner is traveling the fastest? How can you tell?

steeper,

3. Let's see if we can estimate his velocity exactly 30 seconds after leaving the balloon.

$$\frac{f(30-10)-f(30)}{-10}$$

a. What is his average velocity between $t = 20$ and $t = 30$? Show your work.

$$\frac{f(30)-F(20)}{30-20} = \frac{113\,450-121450}{10} = -800 \text{ ft/s}$$

Is this faster or slower than the velocity at exactly 30 seconds? Explain.

slower because the slope between $t=20$ and $t=30$ is flatter than the slope right at $t=30$.

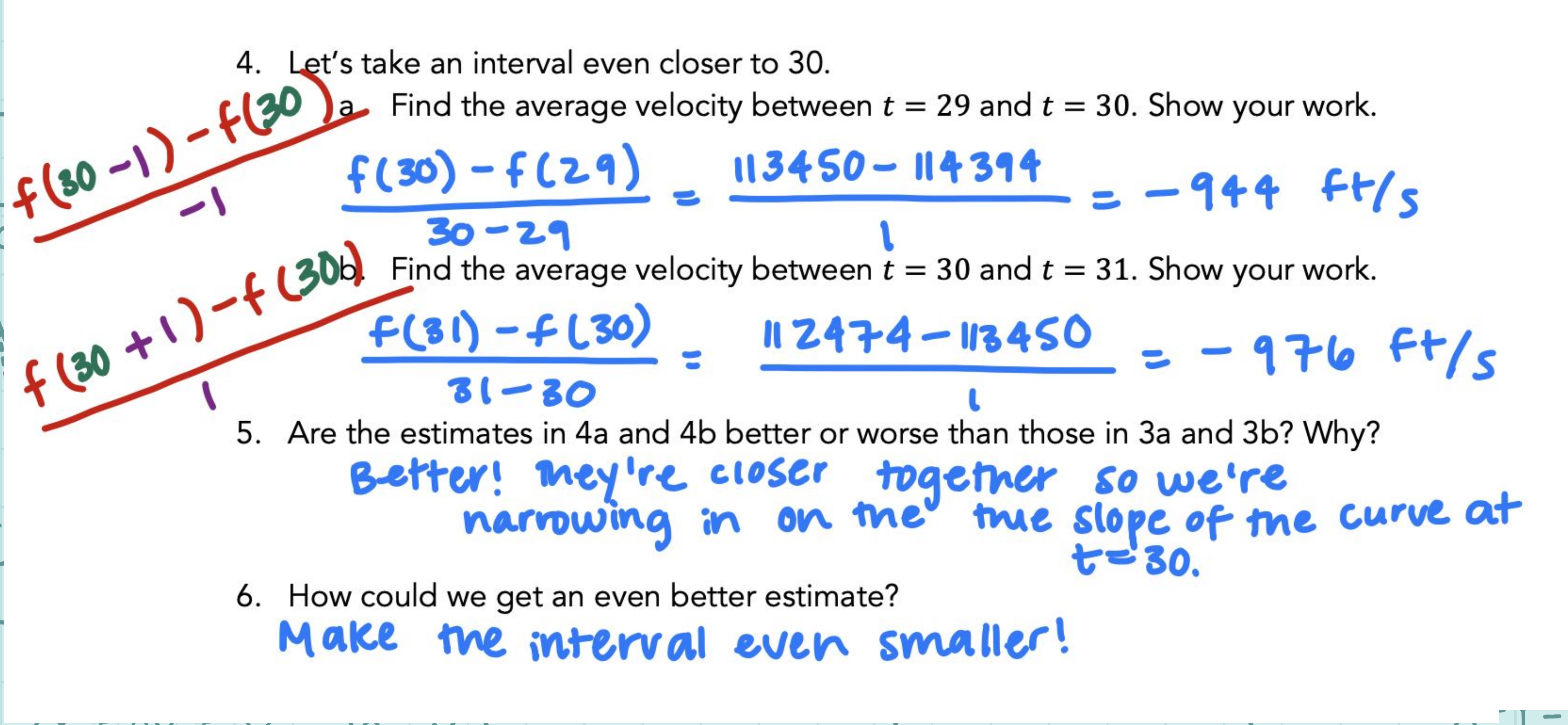
$$\frac{f(30+10)-f(30)}{10}$$

b. What is his average velocity between $t = 30$ and $t = 40$? Show your work.

$$\frac{f(40)-f(30)}{40-30} = \frac{10,2250-113,450}{10} = -1,120 \text{ ft/s}$$

Is this faster or slower than the velocity at exactly 30 seconds? Explain.

faster, the slope gets steeper after $t=30$.



4. Let's take an interval even closer to 30.

   a. Find the average velocity between $t = 29$ and $t = 30$. Show your work.

$$\frac{f(30-1)-f(30)}{-1}$$

$$\frac{f(30)-f(29)}{30-29} = \frac{113450-114394}{1} = -944 \text{ ft/s}$$

   b. Find the average velocity between $t = 30$ and $t = 31$. Show your work.

$$\frac{f(30+1)-f(30)}{1}$$

$$\frac{f(31)-f(30)}{31-30} = \frac{112474-113450}{1} = -976 \text{ ft/s}$$

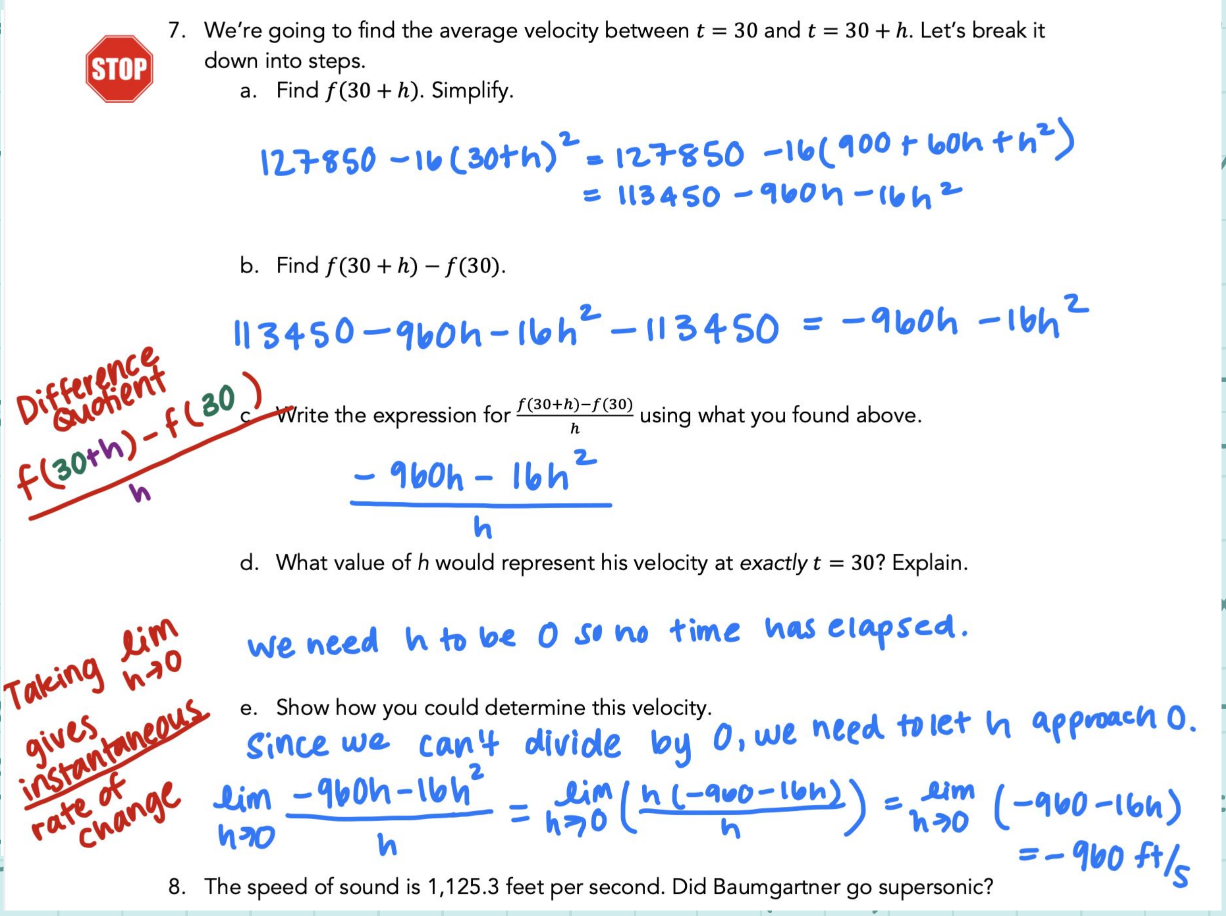
5. Are the estimates in 4a and 4b better or worse than those in 3a and 3b? Why?

Better! They're closer together so we're narrowing in on the true slope of the curve at $t=30$.

6. How could we get an even better estimate?

Make the interval even smaller!

# And now to bring it all home…



7. We're going to find the average velocity between $t = 30$ and $t = 30 + h$. Let's break it down into steps.
   a. Find $f(30 + h)$. Simplify.

$$127850 - 16(30+h)^2 = 127850 - 16(900 + 60h + h^2)$$
$$= 113450 - 960h - 16h^2$$

   b. Find $f(30 + h) - f(30)$.

$$113450 - 960h - 16h^2 - 113450 = -960h - 16h^2$$

Difference Quotient

$$\frac{f(30+h) - f(30)}{h}$$

   c. Write the expression for $\frac{f(30+h)-f(30)}{h}$ using what you found above.

$$\frac{-960h - 16h^2}{h}$$

   d. What value of *h* would represent his velocity at *exactly* $t = 30$? Explain.

we need h to be 0 so no time has elapsed.

Taking $\lim_{h\to0}$ gives instantaneous rate of change

   e. Show how you could determine this velocity.

Since we can't divide by 0, we need to let h approach 0.

$$\lim_{h\to0}\frac{-960h-16h^2}{h} = \lim_{h\to0}\left(\frac{h(-960-16h)}{h}\right) = \lim_{h\to0}(-960-16h)$$
$$= -960 \text{ ft/s}$$

8. The speed of sound is 1,125.3 feet per second. Did Baumgartner go supersonic?

# How to make this transition?

- Opportunities for students to engage with concepts authentically
- Real life contexts to make the mathematics relevant
- Opportunities for students to talk with each other about ideas and to make conjectures
- A safe space where all opinions and thoughts are valued.

# One more thought from Jo on reflection...

"When teachers...replace typical homework questions, which are often not meaningful, with the request to think back on the lesson (reflect) at home, their students report that it increases their mathematical understanding."

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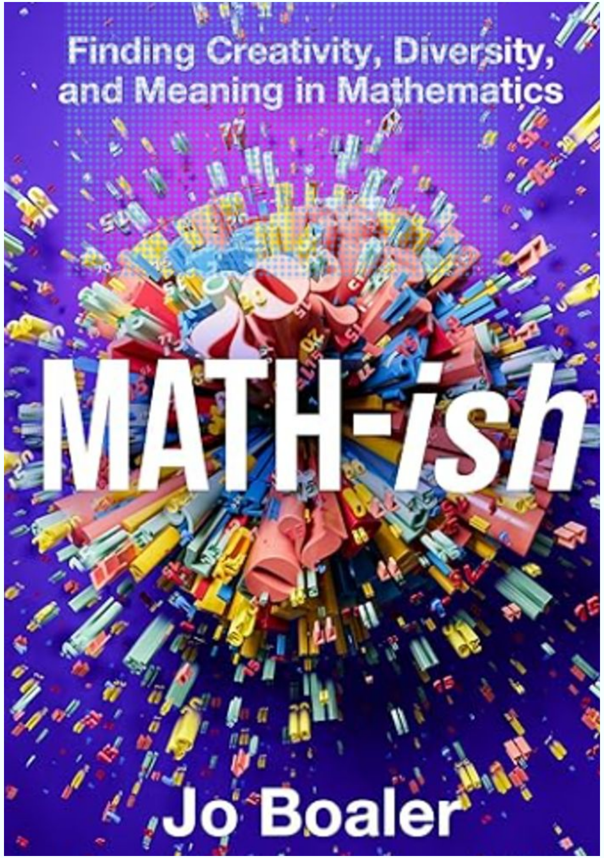
REFLECTION!

Write your answers in the blank spaces below!

What mathematical ideas/concepts did you learn today?

How is the idea you learned today related to others you have learned?



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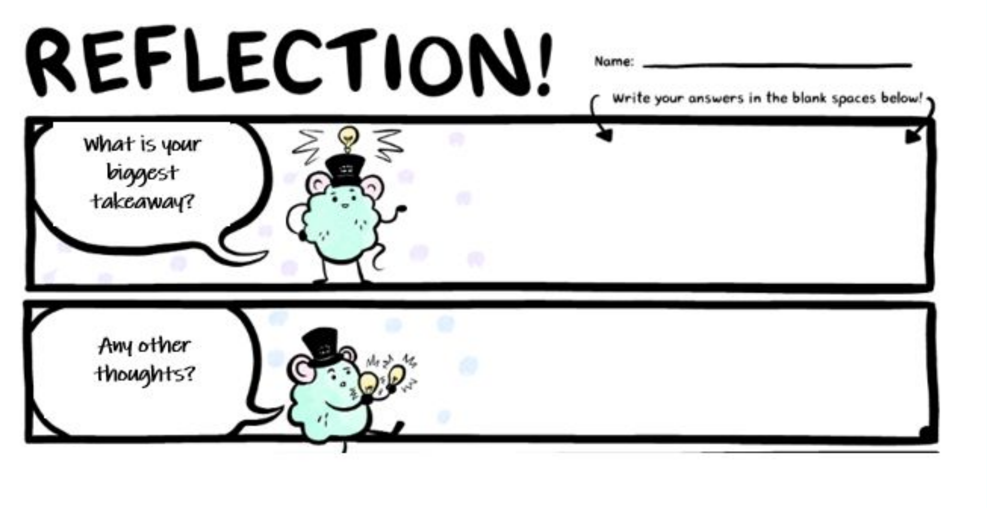
# Now it's your turn!



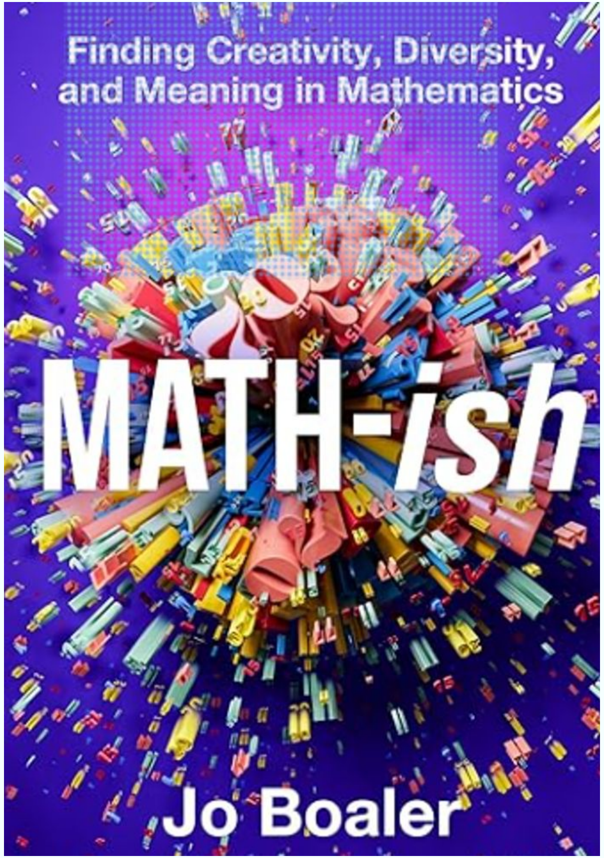
REFLECTION!

Name: ____________

Write your answers in the blank spaces below!

What is your biggest takeaway?

Any other thoughts?



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# Thank you for attending today!

Get materials at
bit.ly/NCTMrateofchange

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