Topic 4.1 Parametric Functions (Daily Video 1)

AP Precalculus

In this video, we will look at how to construct the graph of a parametric function using a table and learn what it means for coordinates to be defined by a parameter.

What are parametric equations?

Notice both number of pumpkins sold and pounds of candy sold are related to the number of days after October 14th, t.

Days after 10/14	Number of Pumpkins Sold	Pounds of Candy Sold
0	2	1
1	3	4
2	4	7
3	5	10
4	6	13

Parametric equations are sets of functions that are linked to the same ______________ variable.

$P(t)$ = number of pumpkins sold

$P(t)$ = ________

$C(t)$ = pounds of candy sold

$C(t)$ = ________

$P(t)$ and $C(t)$ both have the same independent variable, t.

A **parametric function** is a function that has a single input and an output that can be expressed as a set of coordinates. Each coordinate is given by a function of the same variable.

$f(t) = \big(P(t), C(t)\big) =$ (________ , ________)

Calculate $f(2)$. Show your work.

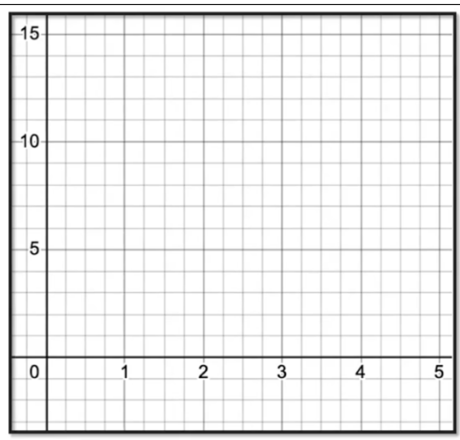
Note: The output consists only of the two dependent variables. The independent variable, which is called the *parameter*, is not part of the coordinates.

Let's look at an Example!

Fill in the table given the following parametric functions and then sketch the graph, labeling the values of t.

$x(t) = t + 1 \qquad y(t) = t^2 + 2t$

t	$x(t)$	$y(t)$	Point
-1			
0			
1			
2			
3			



What should we take away?

- ☐ A parametric function is a function that has a single input and an output that can be expressed as a set of coordinates. Each coordinate is given by a function of the same variable.
- ☐ In order to understand parametric functions, we can construct a table or a graph to help visualize the function.

Topic 4.1 Parametric Functions (Daily Video 2)

AP Precalculus

In this video, we will continue to look at how to construct the graph of a parametric function.

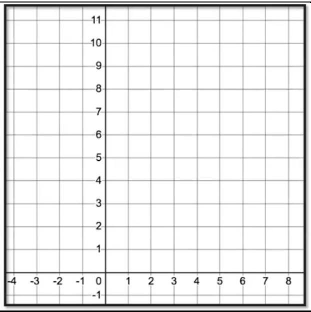
Let's REVIEW!

- ☐ A parametric function is a function that has a single input and an output that can be expressed as a set of coordinates. Each coordinate is given by a function of the ________ ___________. In other words, $f(t) = \big(x(t), y(t)\big)$.
- ☐ The output consists only of the two dependent variable values. The independent variable, which is called the ____________, is not part of the coordinates.

Let's look at an EXAMPLE!

Example 1: Create a table of values for $f(t) = (4 - 2t, 6 - t)$ for $-2 \le t \le 4$. Then, use the table to graph $f(t)$.

t	$x(t)$	$y(t)$	Point
-2			
-1			
0			
1			
2			
3			
4			



Parametric Domain and Direction

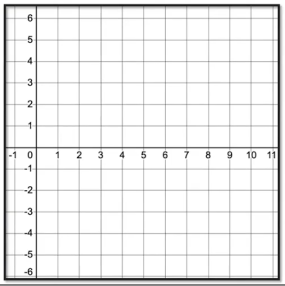
Domain: If the parameter is given an interval, then the parametric function's graph could have a start and an end point. In this example, $-2 \le t \le 4$ means there is a start point of _____ and an end point of ______.

Direction: The graph of a parametric function is NOT always drawn from left to right. In this example, we graphed the function from _________ to ________. Every parametric function follows a direction dictated by the parameter.

Let's PRACTICE!

Example 2: Create a table of values for $f(t) = (4t^2 + 1, 2t)$ for $-1 \le t \le \frac{3}{2}$. Use the table to graph $f(t)$ and note the direction on the graph with arrows.

t	$x(t)$	$y(t)$	Point
-1			
$-\frac{1}{2}$			
0			
$\frac{1}{2}$			
1			
$\frac{3}{2}$			



The graph moves ______ for $-1 < t < 0$. The graph moves _______ for $0 < t < \frac{3}{2}$.

Because the graphs of parametric functions only show outputs, the vertical line test does not apply. Each input has only one output for each coordinate.

What should we take away?

- ☐ The domain of a parametric function is often restricted, which can result in the graph of the function having a start and an end point.
- ☐ The graph of a parametric function has a direction associated with it. Unlike explicit functions, the graph need not be drawn left to right.
- ☐ The vertical line test does NOT apply to graphs of parametric functions.

Topic 4.2 Parametric Functions Modeling Planar Motion (Daily Video 1)

AP Precalculus

In this video, we will explore how parametric equations are related to planar motion, using the position of a particle as an example.

Particle Motion in the Plane

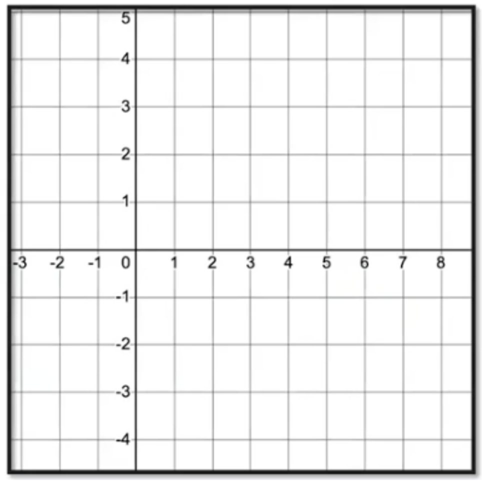
- ____________: a way of stating the location of a particle.
- When a particle is moving in two dimensions, we can give its location by using points in the ______________ coordinate system (the xy-plane).
- All positions depend on where the particle is located in the ____________ direction (x-axis) and the _____________ direction (y-axis).

Let's look at an EXAMPLE!

A particle moves along a curve such that its position at time t is given by $f(t) = (t^2 - 1, 1 - t)$ where $-1 \leq t \leq 3$. Graph the path of the particle.

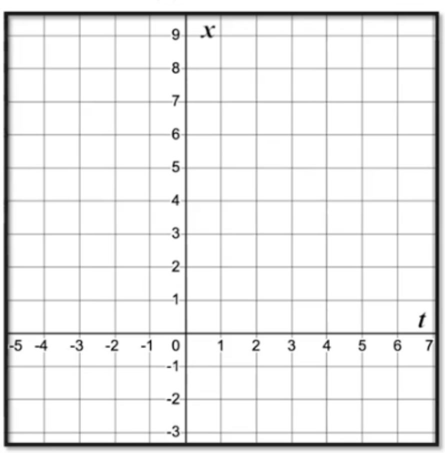
t	$x(t)$	$y(t)$	Point
-1			
0			
1			
2			
3			

Be sure to draw arrows as you plot to the points to show the direction of the particle as it moves.

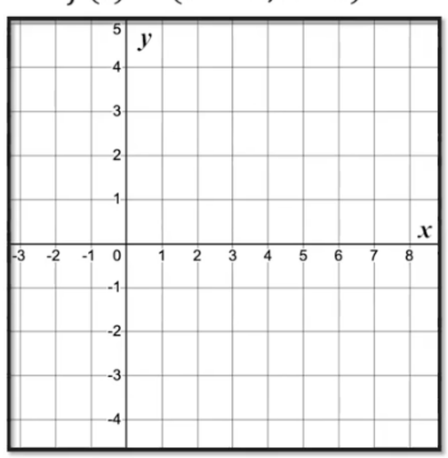


Connecting the Components to the Final Path

$x(t) = t^2 - 1$

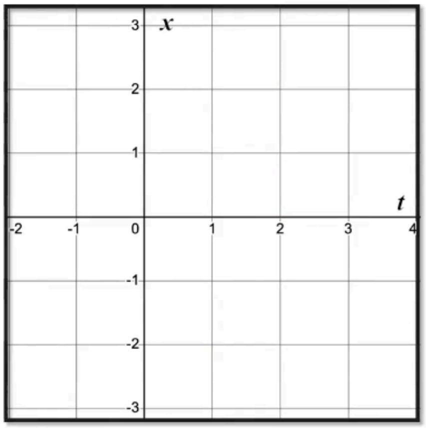


$f(t) = (t^2 - 1, 1 - t)$

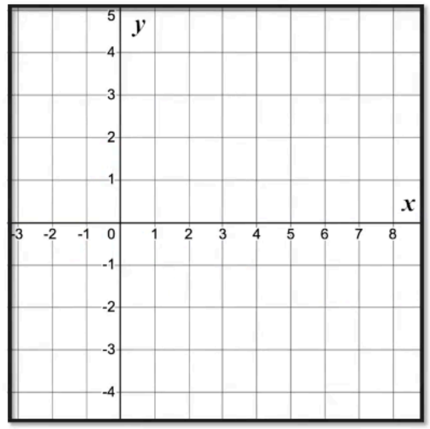


The output of $x(t)$ is the x-coordinate for $f(t)$.

$y(t) = 1 - t$

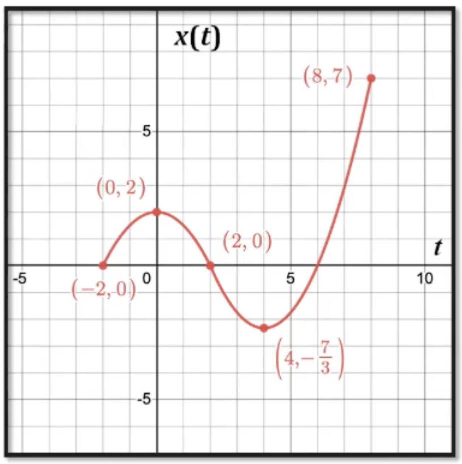
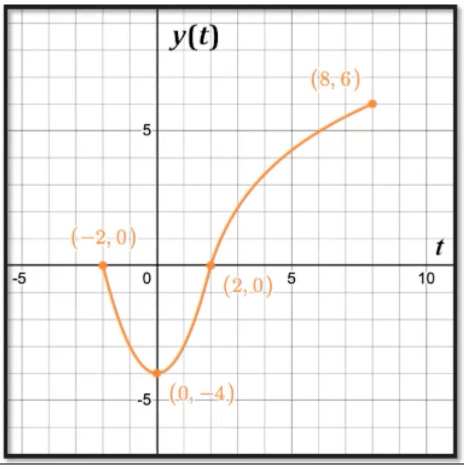


$f(t) = (t^2 - 1, 1 - t)$



The output of $y(t)$ is the y-coordinate for $f(t)$.

Let's PRACTICE!

A particle moves along a curve such that its position is given by $f(t) = (x(t), y(t))$ where $-2 \leq t \leq 8$. Given the graphs of $x(t)$ and $y(t)$ below, what is the farthest right, left, up and down the particle travels. The outputs of $x(t)$ are the ______________ position. Circle the point that corresponds to the maximum value of $x(t)$ and circle the point that corresponds to the minimum value of $x(t)$. Maximum: $x =$ _____ Minimum: $x =$ _____	
The outputs of y(t) are the ______________ position. Circle the point that corresponds to the maximum value of $y(t)$ and circle the point that corresponds to the minimum value of y(t). Maximum: $y =$ _____ Minimum: $y =$ _____	

What should we take away?

- ☐ Parametric functions can be used to model the position of a particle in motion.
- ☐ Horizontal and vertical motion can be examined independent of each other.
- ☐ Use $x(t)$ to answer all questions about horizontal position.
- ☐ Use $y(t)$ to answer all questions about vertical position.

Topic 4.2 Parametric Functions Modeling Planar Motion (Daily Video 2)

AP Precalculus

In this video, we will focus on how the horizontal and vertical components of a particle's motion can be examined independently of each other, which will allow us to find key points on a position graph.

Let's REVIEW!

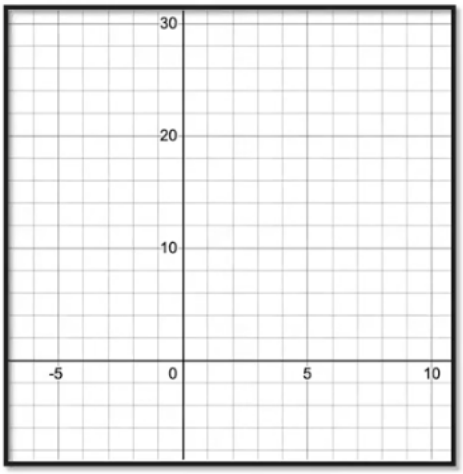
- ☐ Parametric functions can be used to model the position of a particle in motion and allow us to graph the path the particle follows.
- ☐ Horizontal and vertical motion can be examined independent of each other.
- ☐ Use $x(t)$ to answer all questions about ____________ position.
- ☐ Use $y(t)$ to answer all questions about ____________ position.

Let's look at an EXAMPLE!

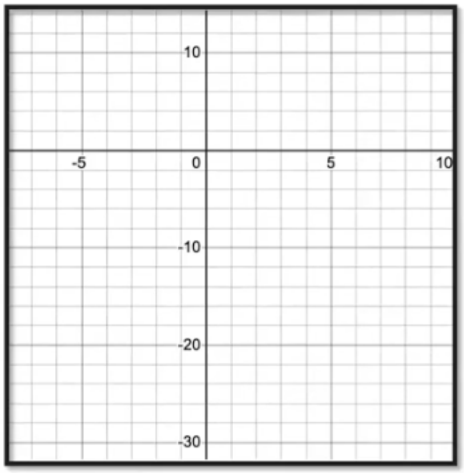
Locate the x-intercepts and y-intercepts for the function $f(t) = (t^2 - 4t, 9 - t^2)$.

x-intercepts $\Rightarrow y = 0$	y-intercepts $\Rightarrow x = 0$
$y(t) = 9 - t^2 = 0$	$x(t) = t^2 - 4t = 0$
The x-intercepts happen when $t =$ ___ and $t =$ ___.	The y-intercepts happen when $t =$ ___ and $t =$ ___.
So…$x(-3) =$	So…$y(0) =$
$x(3) =$	$y(4) =$

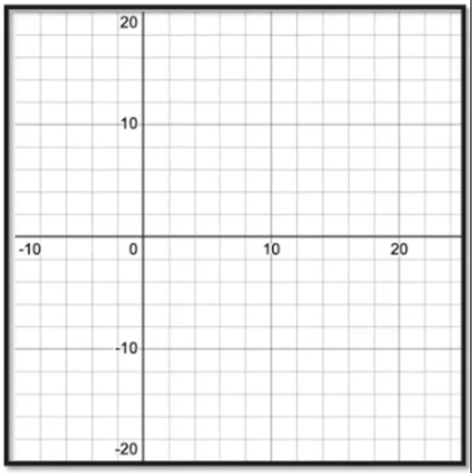
Label the points at $t = -3, 0, 3$ and 4 on each of the three graphs.



$x(t) = t^2 - 4t$



$y(t) = 9 - t^2$



$f(t) = (t^2 - 4t, 9 - t^2)$

What should we take away?

- ☐ Horizontal and vertical motion can be examined independent of each other.
- ☐ Use $x(t)$ to answer all questions about horizontal position.
  - For example: Locating the y-intercepts requires solving $x(t) = 0$.
- ☐ Use $y(t)$ to answer all questions about vertical position.
  - For example: Locating the x-intercepts requires solving $y(t) = 0$.

Topic 4.3 Parametric Functions and Rates of Change (Daily Video 1)

AP Precalculus

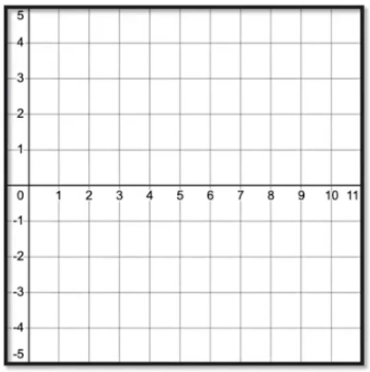
In this video, we will look at how the horizontal and vertical components' dependence on t will help us understand the direction of motion.

Let's REVIEW!

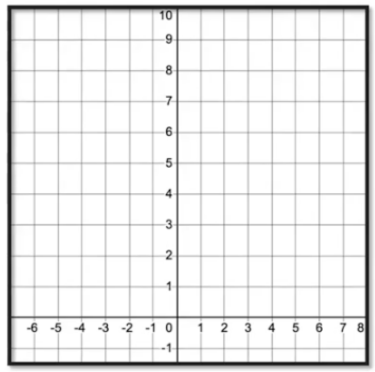
- ☐ Parametric functions can be used to model the position of a particle in motion and to graph the path the particle follows.
- ☐ Use ______ to answer all questions about horizontal position.
- ☐ Use ______ to answer all questions about vertical position.

Examining Horizontal and Vertical Components

Sketch the graphs of $x(t)$ and $f(t)$ below. Annotate to show movement up/down and left/right.



$x(t) = 3\text{sin}t$



$f(t) = (3\text{sin}t, t)$

A particle will be moving…

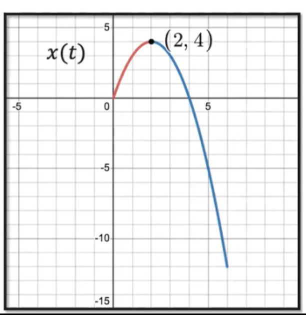
______ if $x(t)$ is increasing ______ if y(t) is increasing

______ if $x(t)$ is decreasing ______ if y(t) is decreasing

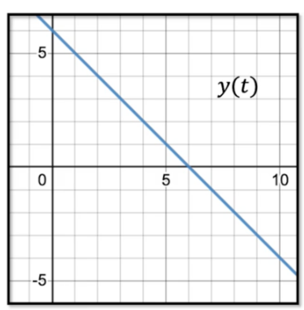
Let's look at an EXAMPLE!

A particle travels with a position given by $f(t) = (4t - t^2, 6 - t)$ for $0 \leq t \leq 6$.

Over what intervals is the particle moving left? Remember: This is when $x(t)$ is decreasing.



Over what intervals is the particle moving up. Remember: This is when y(t) is increasing.



What should we take away?

- ☐ Parametric functions can be used to model the position of a particle and graph its path.
- ☐ A particle will be moving right if $x(t)$ is increasing.
- ☐ A particle will be moving up if y(t) is increasing.
- ☐ A particle will be moving left $x(t)$ is decreasing.
- ☐ A particle will be moving down if y(t) is decreasing.

Topic 4.3 Parametric Functions and Rates of Change (Daily Video 2)

AP Precalculus

In this video, we will look at how the rates of change of a particle's x- and y-coordinates can be used to determine the slope of the particle's path.

Let's REVIEW!

Average Rate of Change

The average rate of change of $f(x)$ for $a \le x \le b$ is given by:

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

This is the same as the two-point slope formula.

The average rates of change of $x(t)$ and $y(t)$ for $t_1 \le x \le t_2$ are given by:

$$\frac{\Delta x}{\Delta t} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} \qquad \frac{\Delta y}{\Delta t} = \frac{y(t_2) - y(t_1)}{t_2 - t_1}$$

Slope of the Path

The slope of the path a particle travels is given by $\frac{\Delta y}{\Delta x}$.

$$\frac{\Delta y}{\Delta x} = \frac{\frac{\Delta y}{\Delta t}}{} = \frac{\frac{ - y(t_1)}{t_2 - t_1}}{\frac{x(t_2) - }{t_2 - t_1}}$$

Let's PRACTICE!

Example 1: Find the slope of the path over the interval $1 \le t \le 9$ if the position is given by $f(t) = \left(t^2 - 5t + 4, 2\sqrt{t} + 5\right)$. Show all your work.

Find $\frac{\Delta y}{\Delta t} =$

Find $\frac{\Delta x}{\Delta t} =$

Find the slope of the path $\frac{\Delta y}{\Delta x} =$

Example 2: Show that $f(t) = \left(t - 2, \frac{1}{2}t - 1\right)$ and $g(t) = \left(2 - \frac{3}{4}t, 1 - \frac{3}{8}t\right)$ follow the same path by finding the slope of the path $\frac{\Delta y}{\Delta x}$, which is the ratio of $\frac{\Delta y}{\Delta t}$ to $\frac{\Delta x}{\Delta t}$.

For $f(t)$: $\frac{\Delta y}{\Delta t} =$ $\qquad \frac{\Delta x}{\Delta t} =$

$\frac{\Delta y}{\Delta x} =$

For $g(t)$: $\frac{\Delta y}{\Delta t} =$ $\qquad \frac{\Delta x}{\Delta t} =$

$\frac{\Delta y}{\Delta x} =$

Two Parameterizations, One Path	
$f(t) = \left(t-2, \frac{1}{2}t-1\right)$	$g(t) = \left(8-t, 4-\frac{1}{2}t\right)$
$x(t) = t-2$ Slope is always ________ So, $x(t)$ is always __________. Particle moves to the ________.	$x(t) = 8-t$ Slope is always ________ So, $x(t)$ is always __________. Particle moves to the ________.
$y(t) = \frac{1}{2}t-1$ Slope is always ________ So, $y(t)$ is always __________. Particle moves _______.	$y(t) = 4-\frac{1}{2}t$ Slope is always ________ So, $y(t)$ is always __________. Particle moves _______.
Both $f(t)$ and $g(t)$ trace out the exact same path, but they do it in __________ directions.	

What should we take away?

- ☐ We can find the average rate of change for the vertical and horizontal individually.
- ☐ The slope of the path is the ratio $\frac{\Delta y}{\Delta x} =$
- ☐ One path can be expressed by many different parameterizations based on the way the particle travels along the path.

Topic 4.4 Parametrically Defined Circles and Lines (Daily Video 1)

AP Precalculus

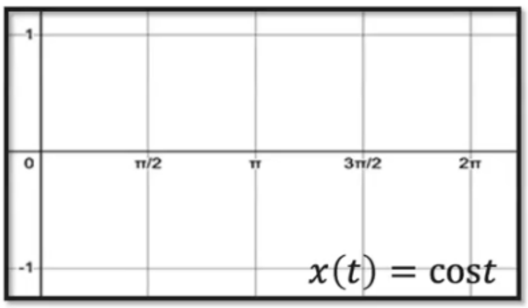
In this video, we will look at how parametric equations can be used to define motion along a circle and a line.

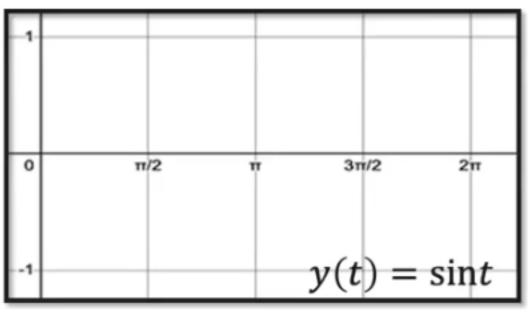
Let's REVIEW!

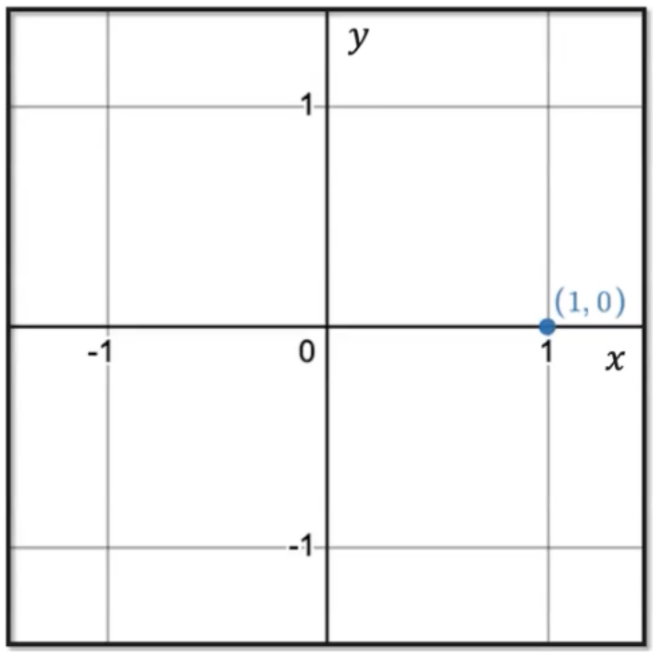
Polar Coordinates Flashback

- ☐ Using polar coordinates, x- and y-coordinates can be expresses as a function of θ.
- ☐ $x = r\cos\theta$ and $y = r\sin\theta$
- ☐ θ is being used as a parameter!
- ☐ $r^2 = x^2 + y^2$

Graphing $f(t) = (\cos t, \sin t)$







Explain why the circle is formed when sketching the graph of $f(t) = (\cos t, \sin t)$.

Let's PRACTICE!

Equations for a Linear Path

Recall the equation of a line:

$$y - y_1 = m(x - x_1)$$

To write the equation of a line, you need…

Point: ________ (also called the initial condition). Slope: $m = \frac{\Delta y}{\Delta x} =$ _______

Example 1: Find a parameterization of the linear path that starts at $(3, -1)$and has a slope of 2.

The initial condition is the starting point.

At $t = 0$, $x(0)=$ _______ and $y(0) =$ ______.

You need $\frac{\Delta y}{\Delta t}$ and $\frac{\Delta y}{\Delta t}$ to be constants with a ratio of ____. This means $\frac{\Delta y}{\Delta t} = 2\left(\frac{\Delta x}{\Delta t}\right)$.

Pick $\frac{\Delta y}{\Delta t} =$ _____ and $\frac{\Delta y}{\Delta t} =$ _____

	$x(0) = 0 + a = 3$	$y(0) = 2(0) + b = -1$
$x(t) = t + a$	$a =$ _____	$b =$_____
$y(t) = 2t + b$	$x(t) =$ __________	$y(t) =$____________

Describe the difference between the graphs of $f(t) = (t + 3, 2t - 1)$ and $g(t) = (2t + 3, 4t - 1)$.

What should we take away?

- ☐ $f(t) = (\cos t, \sin t)$ for $0 \le t \le 2\pi$ is a parameterization of the ______ _________.
- ☐ Any linear path can be parameterized by using an initial condition and keeping

$$m = \frac{\Delta y}{\Delta x} = \frac{\frac{\Delta y}{\Delta t}}{\frac{\Delta y}{\Delta t}} = \text{\_\_\_\_\_\_\_\_\_\_\_}$$

Topic 4.4 Parametrically Defined Circles and Lines (Daily Video 2)

AP Precalculus

In this video, we will look at how knowing the parameterization of a circle at the origin allows us to model any circular path.

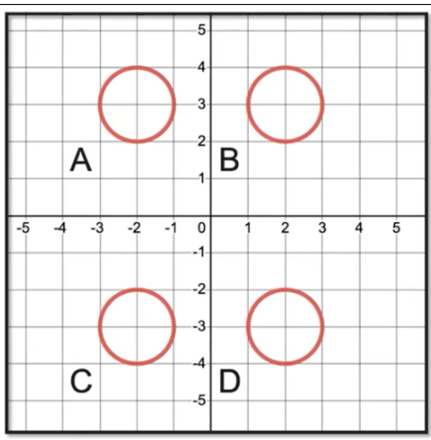
Let's REVIEW!

Translating the Circle

- ☐ $f(t) = (\cos t, \sin t)$ for $0 \le t \le 2\pi$ is a parameterization of the unit circle
- ☐ $x(t) = \cos t$ requires the x-values bounce between -1 and 1 for $0 \le t \le 2\pi$
- ☐ $y(t) = \sin t$ requires the y-values bounce between -1 and 1 for $0 \le t \le 2\pi$
- ☐ $f(t) = (\cos t, \sin t)$ must stay within $-1 \le x \le 1$ and $-1 \le y \le 1$ for $0 \le t \le 2\pi$

Let's look at an EXAMPLE!

Example 1: Which of the following is the graph of $f(t) = (\cos t + 2, \sin t - 3)$?



We know $-1 \le \cos t \le 1$

so…

$\_\_ \le x \le \_\_$

We know $-1 \le \sin t \le 1$

so…

$\_\_ \le y \le \_\_$

The correct answer is _____.

Example 2: Graph $f(t) = (5\cos t, 5\sin t)$

We know $-1 \le \cos t \le 1$

so…

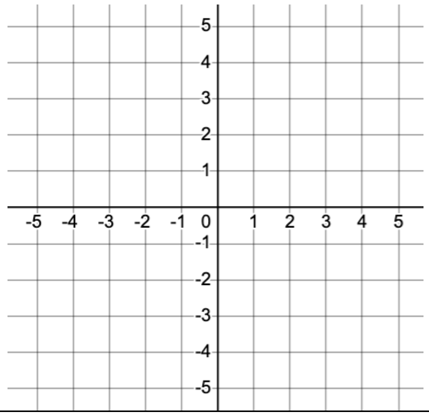
$\_\_ \le x \le \_\_$

We know $-1 \le \sin t \le 1$

so…

$\_\_ \le y \le \_\_$

Graph $f(t) = (5\cos t, 5\sin t)$ below.



Example 3:

Graph $f(t) = (-3 + 2\cos t, 4 + 2\sin t)$

We know $-1 \leq \cos t \leq 1$

so…

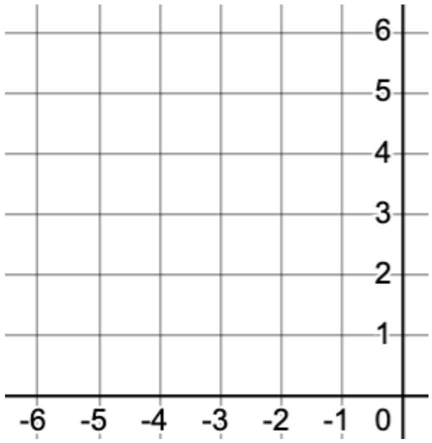
$\_\_ \leq x \leq \_\_$

We know $-1 \leq \sin t \leq 1$

so…

$\_\_ \leq y \leq \_\_$

Graph $f(t) = (-3 + 2\cos t, 4 + 2\sin t)$ below.

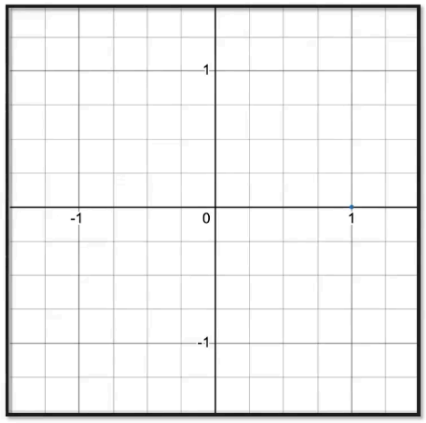


Example 4: Graph $f(t) = (\cos t, -\sin t)$

Why is this paramaterization different than $f(t) = (\cos t, \sin t)$?

Sketch the graph on the right and use an arrow to indicate the direction.

Graph $f(t) = (\cos t, -\sin t)$ below.



What should we take away?

- ☐ $f(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$ is a parameterization of the ______ _________.
- ☐ $f(t) = (a + r\cos t, b + r\sin t)$ is a translation of the unit circle for $0 \leq t \leq 2\pi$.
- ☐ The new circle with have a radius of r.
- ☐ The new circle will have a center of (a, b).

Topic 4.5 Implicitly Defined Functions (Daily Video 1)

AP Precalculus

In this video, we will make graphs for equations that use two variables by finding solutions and defining functions.

Let's REVIEW!

- ☐ A function is a relation that maps a set of ___________ values to a set of _________ values.
- ☐ The function rule $y = f(x)$ shows the relationship between the _____________ variable, x, and the _______________ variable, y.
- ☐ There is exactly _____ value of y for every input of x from the domain of the function.

Implicitly Defined Functions

- ☐ A relation between two ____________.
- ☐ An implicitly defined function is the set of solutions to an equation involving both variables.
- ☐ The graph of an implicitly defined function displays the _____________ to the equation.
- ☐ There may be ____________ values of y that satisfy the relationship for each value of x, and vice versa.
- ☐ Solving for one variable can define a function for all or part of its implicitly defined graph.

Let's look at an EXAMPLE!

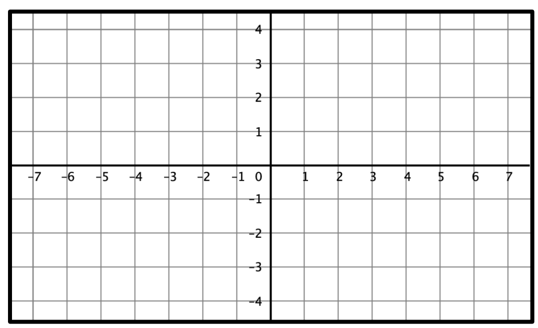
Example 1: $2x - 5y = 3$

Rewrite the equation as a function of x:

Rewrite the equation as a function of y:

Note: Each of these functions defines the entire graph that displays the implicit equations.

Sketch the graph of the function below.



Example 2: $x^2 + y^2 = 4$

This is a circle with a radius of ____.

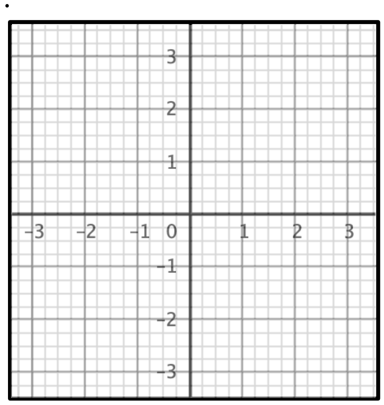
Why is this NOT a function?

Solve for y to get two functions:

$y =$ ___________ $y =$ ___________

Each of these functions defines a graph that is part of the implicit graph.

Sketch the graph of the function below.



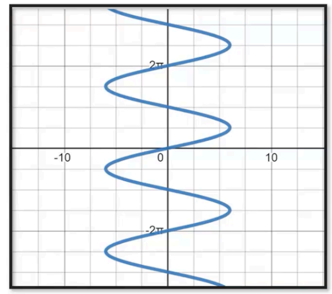
Example 3: $x = 6 \sin y$

This is a function of y instead of x.
$x = f(y) =$ ___________

Use inverse functions to solve for y:
$y =$ __________

We can transform the function of y to produce graphs that define other intervals of the implicitly defined graph.

Shade a piece of the graph that would result in a function and pass the vertical line test.

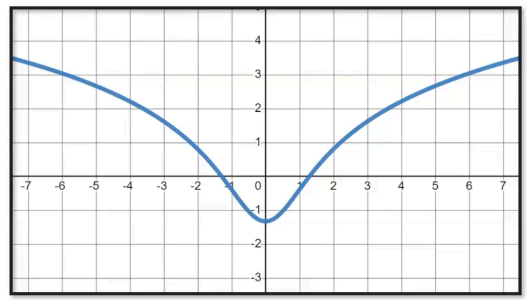


Example 4: $y + 5e^{y} = 3x^{2}$

The graph of this implicitly defined function shows that there is only one y ________ value for each value of x____________.

Solve the relation for x to produce two functions:

$x =$ __________________



Example 5: $x^{3} + y^{3} = 12xy$

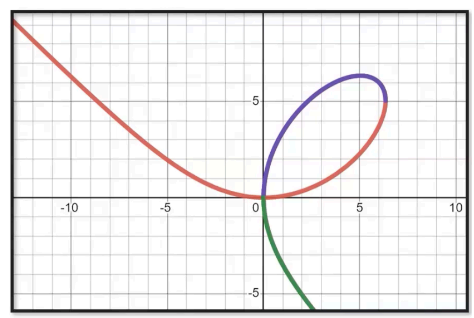
This equation cannot be easily solved for x or y.

A computer algebra system can be used to solve for the three functions of x needed to create this graph.

However, you can also plot points to create the graph.

Show the solution for y, when $x = 0$.

Show the solution for y, when $x = 1$.



What should we take away?

- ☐ Implicitly defined functions are solutions to equations of ______ variables.
- ☐ The graph of an implicit relation shows all the ___________ to the equation.
- ☐ Some implicit equations can be solved as a function of one ____________.
- ☐ For other implicit equations, a function of one variable can define a graph that is ________ of the original solution set.

Topic 4.5 Implicitly Defined Functions (Daily Video 2)

AP Precalculus

In this video, we will explore the rates of change of x with respect to y and y with respect to x for implicitly defined functions.

Let's REVIEW!

- ☐ A graph of a function shows how the function's input and output values _________.
- ☐ The rate of change at a point can be approximated by the __________ rates of changes of the function over ________ intervals containing the point.

Using Implicitly Defined Functions

- ☐ Using points on an implicit graph that are _______ to one another can be used to determine how the two quantities vary together.
- ☐ When the ratio of change is _____________, the two variables either ________ increase or both decrease.
- ☐ When the ratio of change is ______________, as one variable increases, the other variable _______________.
- ☐ When the rate of change of x with respect to y is zero, the graph has a ___________ interval.
- ☐ When the rate of change of y with respect to x is zero, the graph has a ___________ interval.

Let's look at an EXAMPLE!

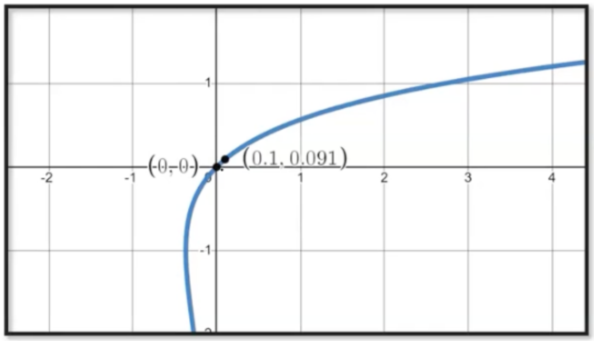
Example 1: $y \cdot e^y = x$ near $x = 0$

The point _______ is a solution to the equation.

When $x = 0.1$, solving for y leads to ________.

The ratio of the change is:

Because the ratio is ________, the two variables both increase or both decrease near this point.



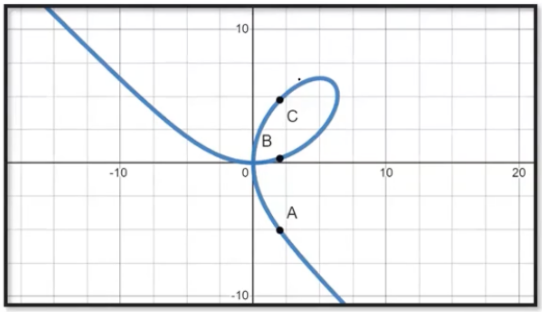
Example 2: $x^3 + y^3 = 12xy$ near $x = 2$

There are three points on the graph with an x-coordinate of 2.

The ratio of change at each point is:

- Point A: $\frac{-5.195-(-5.058)}{2.1-2} =$
- Point B: $\frac{0.370-0.335}{2.1-2} =$
- Point C: $\frac{4.825-4.723}{2.1-2} =$

As one variable increases, the other variable ____________ near point A.



The two variables simultaneously ____________ or __________ near points B and C.

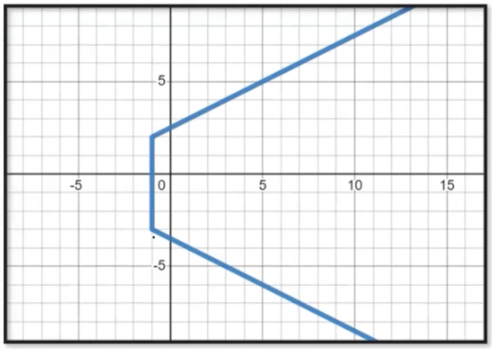
Example 3: $x + 6 - |y - 2| = |y + 3|$

This equation is piecewise-defined for intervals of y.

For $-3 \leq y \leq 2$, the equation is equivalent to

______________________________ or $x =$ ______.

Because the value of x is constant on this interval, the rate of change of x with respect to y is ________. This means the graph is vertical on $-3 \leq y \leq 2$.



Let's PRACTICE!

An implicitly defined curve has points at $(2.38, 4.72)$ and $(P, 4.63)$. The ratio of the change in the two variables is negative. Which of the following statements about the value of P must be true?

A) $P > 0$
B) $P < 0$
C) $P > 2.38$
D) $P < 2.38$

What should we take away?

- ☐ Using points on the graph of an implicitly defined function that are near one another can be used to determine how the two quantities vary together.
- ☐ Implicitly defined functions can have intervlas that are _______________ or _____________.

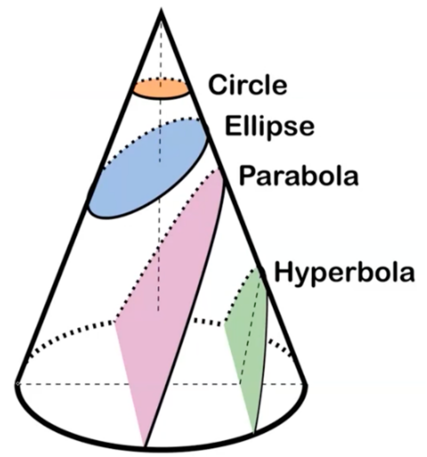
Topic 4.6 Conic Sections (Daily Video 1)

AP Precalculus

In this video, we will learn about conic sections formed by intersections of a plan and a cone, as well as how to identify which conic section matches each standard equation.

Conic Sections

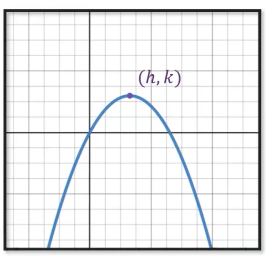
- ☐ Conic sections are ________________ formed by the intersection of a plane and the surface of a cone.
- ☐ If the slice of the plane is parallel to the circular face of the cone, the conic section is a __________.
- ☐ If the slice of the plane is perpendicular to the circular face of the cone, the conic section is a __________.
- ☐ If the slice of the plane is parallel to the circular face of the cone, the conic section is a __________.
- ☐ If the plane is not parallel or perpendicular, an ________________ or a ______________ is formed.
- ☐ If the plane is parallel to the edge of the cone, a _______________ is formed.



Parabola with Vertex (h, k)

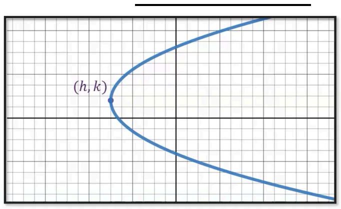
Opens up or down

$y - k =$ ________



Opens left or right

$x - h =$ ____________



Draw the axis of symmetry on each parabola.

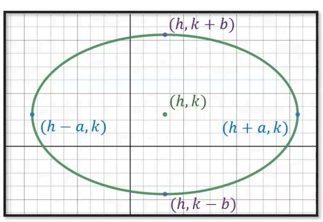
Ellipse with Center (h, k)

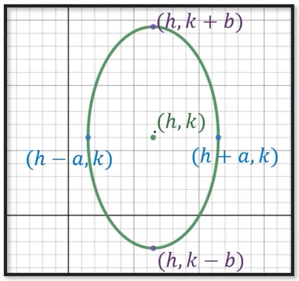
Horizontal when a__b

Vertical when a__b

Circle when a__b

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = \_\_\_$$





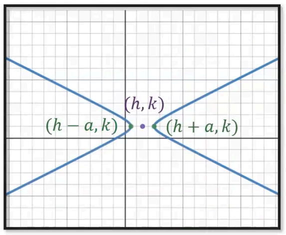
Label distances of a and b from the center (h, k).

The _________ axis is longer and the ______ axis is shorter.

Hyperbola with Center (h, k)

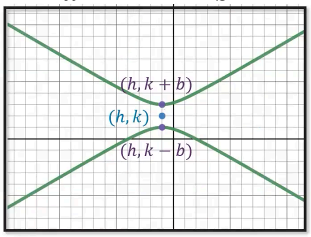
Opens ________________

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

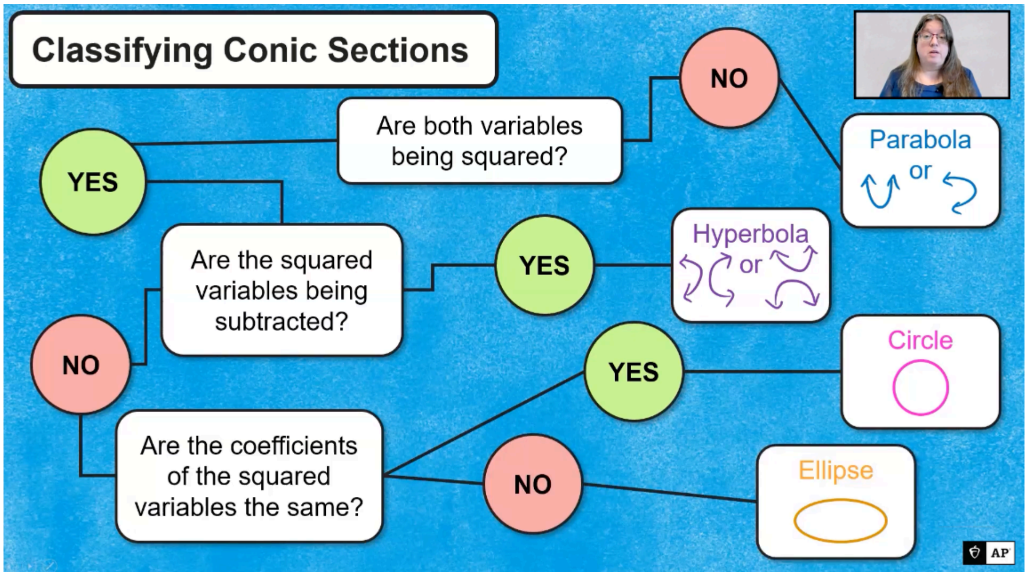


Opens ________________

$$-\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



Sketch the slant asymptotes on each hyperbola.



Let's PRACTICE!

MCQ1: Which conic section is represented by the equation $3x^2 - 7y^2 + 2x + 3y - 5 = 0$? A) Circle B) Ellipse C) Hyperbola D) Parabola	MCQ2: Which conic section is represented by the equation $2y^2 - 6x + y + 1 = 0$? A) Circle B) Ellipse C) Hyperbola D) Parabola	MCQ3: Which conic section is represented by the equation $4y^2 - 3y + 1 = -2x^2 + x + 3$? A) Circle B) Ellipse C) Hyperbola D) Parabola

What should we take away?

- ☐ Implicitly defined functions called conic sections are cross sections of a _________.
- ☐ We can identify which type of conic is defined by examining the degree 2 variables:
  - o If only x or y is squared, the conic is a _______________.
  - o If both x and y are squared and the squared variable terms are <u>added</u>, the conic is an _____________.
  - o If both x and y are squared and the squared variable terms are <u>subtracted</u>, the conic is an _____________.
- ☐ The circle is a special case of the ellipse that occurse when <u>both</u> squared variables have the same coefficient.

Topic 4.6 Conic Sections (Daily Video 2)

AP Precalculus

In this video, we will express conic sections in standard form and construct their graphical representations.

Let's look at an EXAMPLE!

Example 1: Graph the conic section given by: $4y + (x + 1)^2 = 8$

Rearrange the terms $4y - 8 =$ ________

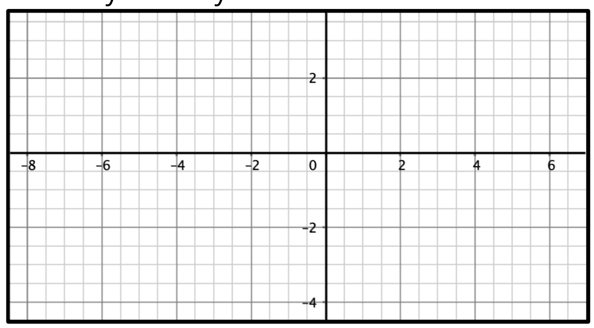
Factor the left side ___________ = __________

Write in standard form. $y - 2 =$ _________

We Know:

- ☐ The vertex is at _______
- ☐ The axis of symmetry is _______
- ☐ The parabola opens ____________

Sketch the graph and label the vertex and axis of symmetry.



Example 2: Graph the conic section given by: $(y - 3)^2 = x + 4$

We Know:

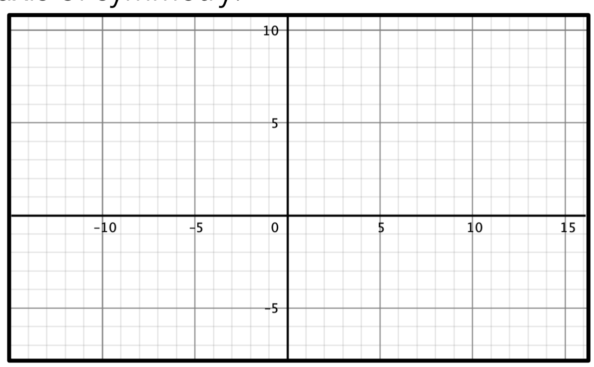
- ☐ The conic section is a parabola which opens to the _________ or ________
- ☐ The vertex is at _______
- ☐ The axis of symmetry is _______
- ☐ The parabola opens to the ______

When $y = 0$, then $x =$ _______

When $y = 1$, then $x =$ _______

Then, use the axis of symmetry to find other points on the parabola.

Sketch the graph and label the vertex and axis of symmetry.



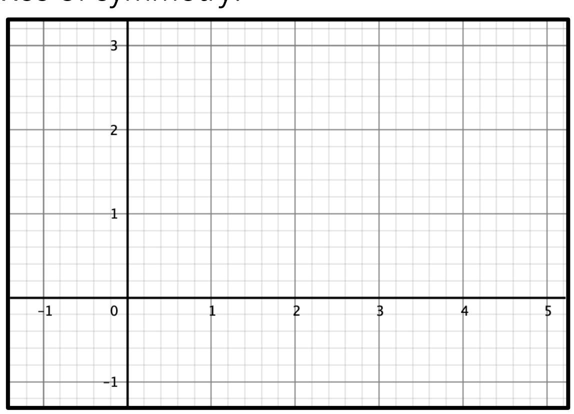
Example 3: Graph the conic section given by: $4(x - 2)^2 + 9(y - 1)^2 = 36$

Write in standard form:

We Know:

- ☐ The conic section is an ____________
- ☐ The center is at ________.
- ☐ Axes of symmetry are _______ and _______
- ☐ Horizontal radius is ____, vertical radius is ____

Sketch the graph and label the center and axes of symmetry.



Example 4: Graph the conic section given by: $\frac{(y-1)^2}{2^2} - \frac{(x+3)^2}{4^2} = 1$

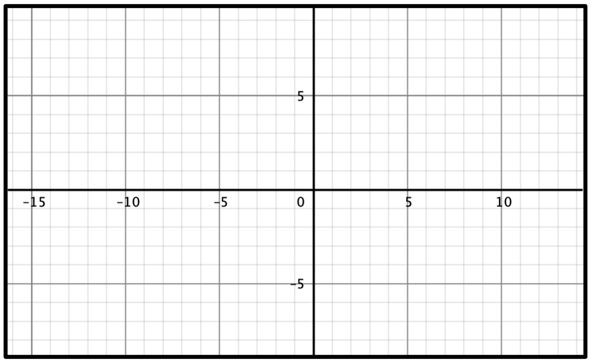
We Know:

- ☐ The graph is a ____________
- ☐ The center is at _______
- ☐ The vertices are _______ and _______
- ☐ The axis of symmetry are ______ and ________

The values of a and b help define the vertices and define the slopes of the asymptotes:

$$y - k =$$

Sketch the graph and label the center, axes of symmetry and asymptotes.



Multiple-Choice Practice: Which of the following best describes the hyperbola:

$$25(x-1)^2 - 4(y-3)^2 = 100?$$

A) Opens left and right with vertices of $(-6, 3)$ and $(4, 3)$

B) Opens left and right with vertices of $(-3, 3)$ and $(1, 3)$

C) Opens up and down with vertices of $(-1, -2)$ and $(-1, 8)$

D) Opens up and down with vertices of $(-1, 1)$ and $(-1, 5)$

Solution:

Rewrite the equation in standard form:

How do you know the direction it opens?

The center is _______ and the vertices are ___________.

What should we take away?

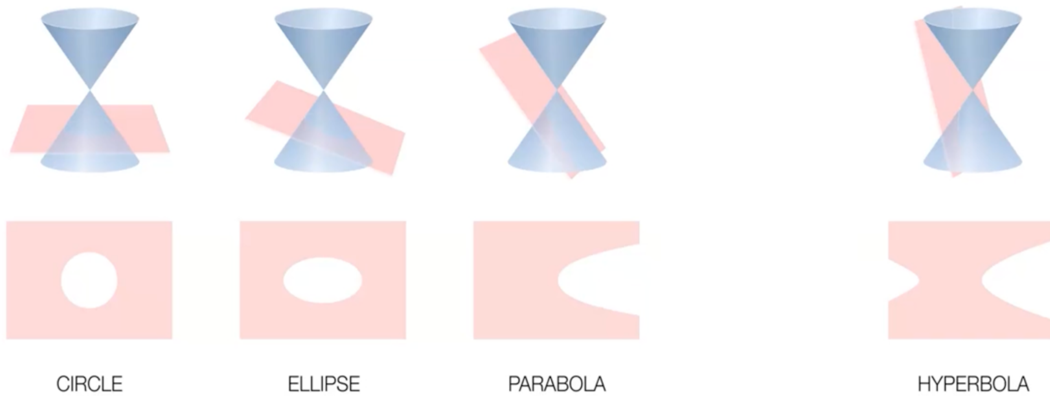
- ☐ When the equations for conic sections are written in standard form, we have a lot of information about the shape of the related graph and its symmetries.
- ☐ We can use the standard form equations to construct graphs of conic sections.

Topic 4.6 Conic Sections (Daily Video 3)

AP Precalculus

In this video, we will connect key features of conic section equations with their graphs.

Let's REVIEW!



CIRCLE	ELLIPSE	PARABOLA	HYPERBOLA
$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$		$y-k=a(x-h)^2$ $x-h=a(y-k)^2$	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Let's look at an EXAMPLE!

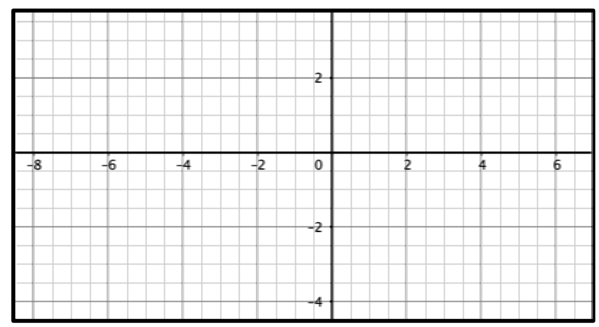
Example 1: Write an implicitly defined function for the parabola with vertex $(4,-1)$, axis of symmetry $y=-1$ and x-intercept of 2.

Start with a sketch of the information.

How do you know the parabola opens to the left?

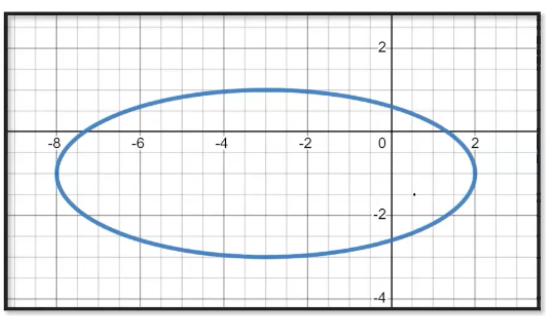
Substitute values into the form: $x-h=a(y-k)^2$
Using the given vertex:

Substitute the x-intercept, (2, 0) to find the value of a.



Example 2: Write an implicitly defined function for the ellipse given in the graph below:

Sketch the center and the label the lengths of the horizontal radius and vertical radius.



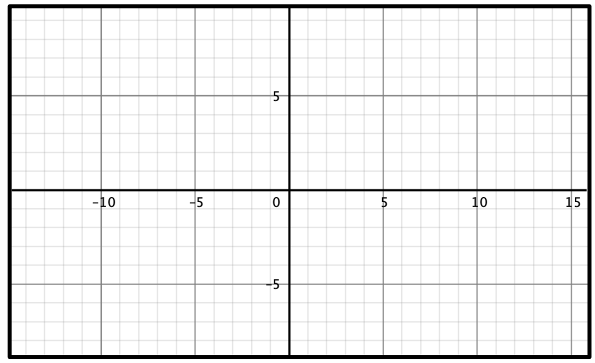
The general form of the equation is:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Substitute the value of the center $(-3,-1)$ and the values for the horizontal and vertical radii.

Example 3: Write an implicitly defined function for the hyperbola with asymptotes of $y + 2 = \pm\frac{1}{3}x$ that passes through the point $(6, -2)$.

Sketch the asymptotes and the given point.



The point of intersection of the asymptotes is _____.

The hyperbola will open to the ______ and ______ and have an equation of the form:

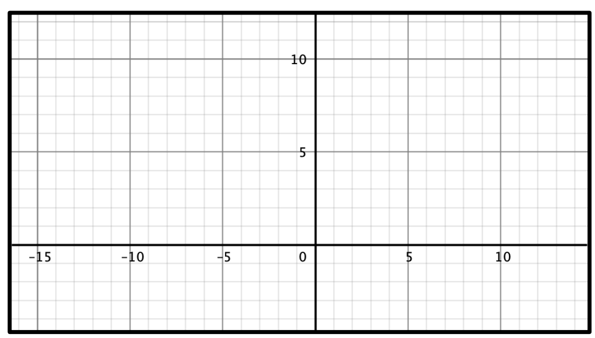
Since $(6, -2)$ is 6 units to the right of the center, the value of $a =$_____.

Use the slope of the slant asymptote to find b:

The final equation is:

Example 4: Consider the conic section with axes of symmetry $y = 5$ and $x = -3$, which passes through the points $(-3, 1)$ and $(1, 5)$. Identify the conic and write an equation to represent its graph.

Make a sketch of the axes of symmetry.



It is NOT a parabola, because ____________________________.

It is NOT a hyperbola, because ____________________________.

The conic must be ______________.

The axes of symmetry intersect at ________.
The length of the horizontal radius is ____.
The length of the vertical radius is ____.
Both radii are the same length. So, this is a _______.
The equation is:

What should we take away?

Information about the center, vertex, lines of symmetry, and asymptotes of the graph of a conic section allows you to translate between the graph and its implicit equation.

Topic 4.7 Parametrization of Implicitly Defined Functions (Daily Video 1)

AP Precalculus

In this video, we will combine the ideas of parametric equations and implicitly defined functions to write parametrizations of equations involving two variables.

Let's look at an EXAMPLE!

Example 1: Verify that $(t^2, 1-t^3)$ is a parametrization for $x^3 = (1-y)^2$	Example 2: Write a parametrization for the function $e^x \cdot y = x^2 \cdot y + 3$
Substitute the parametric expressions into the implicit function: $x^3 = (1-y)^2$ $(t^2)^3 =$ ________ $(t^2)^3 =$ ________ $(t^2)^3 =$ ________ ____ = ____ Note the graphs of the parameterization and the implicit function are the same graph!	Rearrange the equation and factor to solve for y: $e^x \cdot y = x^2 \cdot y + 3$ $e^x \cdot y - x^2 \cdot y =$ ____ (________)$\cdot y = 3$ $y =$ ________ The function (____, ______) will parametrize the curve.

Example 3: Write parametrizations for the function $e^x \cdot y = 3$ and its inverse.	
Solving for y yields: $y = f(x) =$ ________ Which leads to the parametrization: $(t, f(t)) =$ (____, ______)	Because this function is invertible, we can write a parameterization for its inverse: $(f(t), t) =$ (______, ____)

What should we take away?

- ☐ When a set of parametric equations corresponds to an implicitly defined function, every point on the parametrization will be a __________ to the implicit equation.
- ☐ When an implicitly defined function can be rewritten in the form $y = f(x)$, the function can be parametrized as __________.
  - o When this function is invertible, the inverse can be parametrized as __________ for an appropriate domain of t.

Topic 4.7 Parametrization of Implicitly Defined Functions (Daily Video 2)

AP Precalculus

In this video, we will determine parametric versions of the conic section equations.

Let's look at an EXAMPLE!

Parametrizing Parabolas Example:

Find a parametric function for the parabola: $(y-3)^2 = \frac{1}{2}(x+1)$

We can solve this equation for x: __________ $= (x+1)$

______________ $= x$

Set $y = t$ and substitute: (______________, _____)

Note: Every parabola can be expressed as a function of either x or y, so the method shown at the left will always find a parametrization.

Let's REVIEW!

We can use the Pythagorean identities: ____________________ and __________________ along with the implicit equations for an ellipse and a hyperbola, to write parametric functions for these curves.

Parametrizing Ellipses

Claim: $\left(x(t), y(t)\right) = (h + a\cos t, k + b\sin t)$ is a parametrization for an ellipse centered at (h, k) with horizontal radius a and vertical radius b.

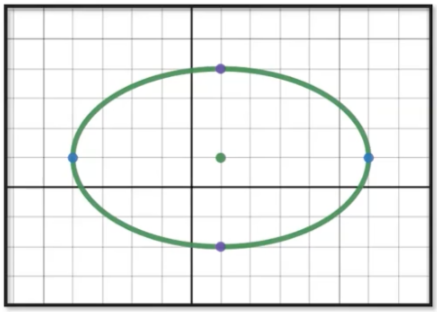
Substitute the parametric equations into the implicit equations for an ellipse to verify the claim.

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(\_\_\_\_\_\_\_\_\_\_ - h)^2}{a^2} + \frac{(\_\_\_\_\_\_\_\_\_\_ - k)^2}{b^2} = 1$$

$$(\cos t)^2 + \_\_\_\_\_\_\_\_\_\_ = 1$$

Label the center and horizontal radius and vertical radius on the graph.



Parametrizing Hyperbolas

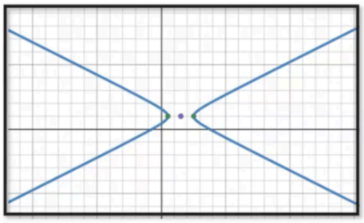
Claim: Hyperbolas can be parametrized by trigonometric functions for $0 \leq t \leq 2\pi$.

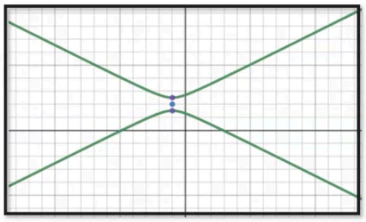
Opening left and right:

$\left(x(t), y(t)\right) = (h + a\sec t, k + b\tan t)$

Opening up and down:

$\left(x(t), y(t)\right) = (h + a\tan t, k + b\sec t)$





Let's look at an EXAMPLE!

Find the parametrization for the hyperbola:

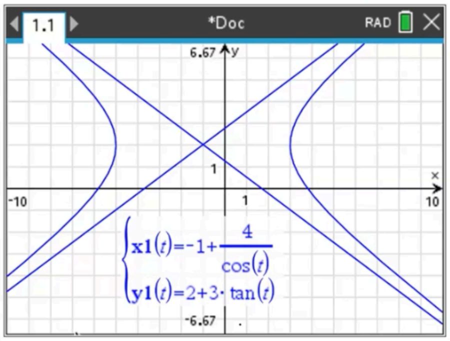
$$-\frac{(x+1)^2}{16}+\frac{(y-2)^2}{9}=1$$

This hyperbola opens left and right with a center of ________.

$x(t) =$ ________________

$y(t) =$ ________________ for $0 \le t \le 2\pi$

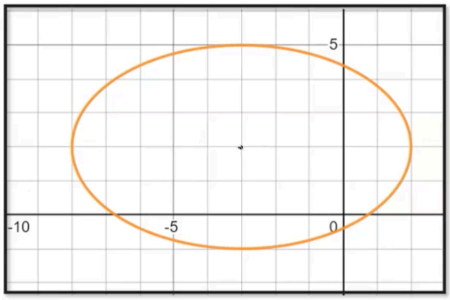
An advantage of parametric form is graphing on a handheld calculator.



Let's PRACTICE!

Multiple Choice Example

Which equation can be used to model the ellipse below for $0 \le t \le 2\pi$.



A) $(2+5\cos t, -3+3\sin t)$

B) $(2+3\cos t, -3+5\sin t)$

C) $(-3+5\cos t, 2+3\sin t)$

D) $(-3+10\cos t, 2+6\sin t)$

Explain why answer choices A and B can be eliminated.

Describe the error a student might be making if they choose choice D.

What should we take away?

Parabolas can be parametrized by solving for x or y.

- When solving for x, replace y with t and write $(x(t), y(t)) = (t, f(t))$
- When solving for y, replace x with t and write $(x(t), y(t)) = (f(t), t)$

Ellipses can be parameterized using _______ and ________ for $0 \le t \le 2\pi$ by $(x(t), y(t)) = (h + a\cos t, k + b\sin t)$.

Hyperbolas can be parameterized using ________ and __________ for $0 \le t \le 2\pi$ by
When hyperbolas open up and down, $(x(t), y(t)) = (h + a\tan t, k + b\sec t)$.
When hyperbolas open left and right, $(x(t), y(t)) = (h + a\sec t, k + b\tan t)$.

Topic 4.8 Vectors (Daily Video 1)

AP Precalculus

In this video, we will learn about different characteristics of vectors.

What is a vector?

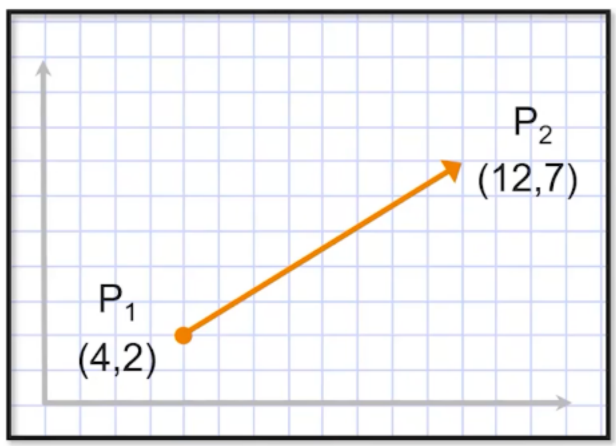
- A ________________ quantity is one that can be represented by a single real number, or a magnitude.
- A ________________ is a quantity that has both ______________ AND ______________.

Let's PRACTICE!

Scalar vs Vector Quantities

Driving 25 miles per hour in my neighborhood	**Scalar Quantity**	**Vector Quantity**
Driving 60 miles per hour due east on the highway	**Scalar Quantity**	**Vector Quantity**
The number of apples I've eaten today	**Scalar Quantity**	**Vector Quantity**

Characteristics of a Vector



A vector is a directed line segment.

The point at the beginning is called the ___________.

The point at the end is called the ___________.

Component Form: $\vec{\boldsymbol{v}} = \langle \boldsymbol{a}, \boldsymbol{b} \rangle$

a is called the x −_____________ or horizontal component. Show a graphically on the grid to the right.

b is called the y −_____________ or vertical component. Show b graphically on the grid to the right.

Calculate the components of $\overrightarrow{P_1P_2}$. Show all work. $\overrightarrow{P_1P_2} = \langle$______ , ______$\rangle = \langle$______ , ______$\rangle$.

Magnitude: The length of the directed line segment, denoted $|\vec{v}|$. Calculate the magnitude of the vector $\overrightarrow{P_1P_2}$. Show all work. $|\vec{v}| =$

The magnitude of $\vec{v} = \langle a, b \rangle$ is calculated using the _______________ Theorem. $|\vec{v}| =$ ___________

Finding the Components

Draw x and y −components of the vector $\vec{w}$ on the graph to the right.

Calculate the x −component of $\vec{w}$. Show all work.

$\cos(\theta) = \_\_\_\_\_\_\_\_$

Calculate the y −component of $\vec{w}$. Show all work.

What should we take away?

A vector is a quantity that has both _______________ and _______________.

Magnitude: Given vector: $\vec{v} = \langle a, b \rangle$

$$|\vec{v}| = \sqrt{a^2 + b^2}$$

We can solve for the components of a vector given the vector's magnitude and angle using right triangle trigonometry.

Topic 4.8 Vectors (Daily Video 2)

AP Precalculus

In this video, we will focus on vector operations and unit vectors.

What should I remember ?

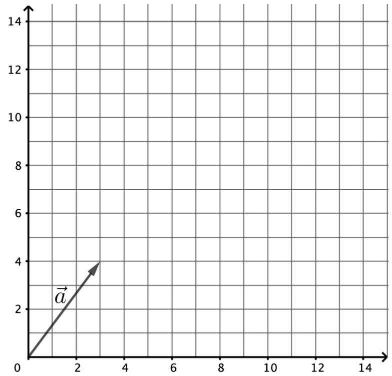
A vector is a quantity that has both
_______________ AND ________________.

> **Magnitude:** Given vector: $\vec{v} = \langle a, b \rangle$
>
> $|\vec{v}| = \sqrt{a^2 + b^2}$

Scalar Multiplication

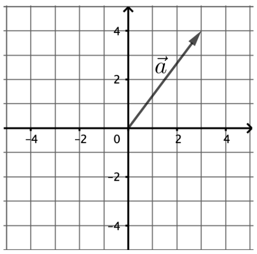
$\vec{a} = \langle 3,4 \rangle \qquad 3\vec{a} = \langle \qquad , \qquad \rangle$

Graph $3\vec{a}$ with tail at the origin on the grid to the right.

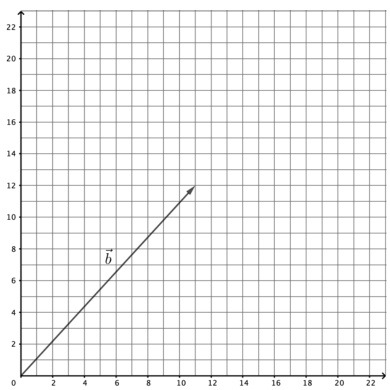


Calculate the components of $-\vec{a} = \langle \qquad , \qquad \rangle$

Graph $-\vec{a}$ with its tail at the origin on the grid below.



Vector Addition and Subtraction



Given the vectors $\vec{b} = \langle 11,12 \rangle$ and $\vec{c} = \langle 9,3 \rangle$, show the addition of $\vec{b} + \vec{c}$ graphically on the grid to the left.

Graph the resultant vector $\vec{b} + \vec{c}$ from the tail of $\vec{b}$ to the head of $\vec{c}$.

What are the components of $\vec{b} + \vec{c}$? $\quad \vec{b} + \vec{c} = \langle \qquad , \qquad \rangle$

Show analytically that you get the same components for $\vec{b} + \vec{c}$. Show your work. $\qquad \vec{b} + \vec{c} =$

Show the subtraction of $\vec{b} - \vec{c}$ graphically on the grid above and verify analytically that you get the same components for $\vec{b} - \vec{c}$. Show your work. $\qquad \vec{b} - \vec{c} =$

Let's PRACTICE!

Given the vectors $\vec{r} = \langle -3,2 \rangle$, $\vec{s} = \langle 4,0 \rangle$, and $\vec{t} = \langle 5,6 \rangle$, evaluate the following. Show your work.

1) $-4\vec{r}$

2) $\vec{s} + \vec{t}$

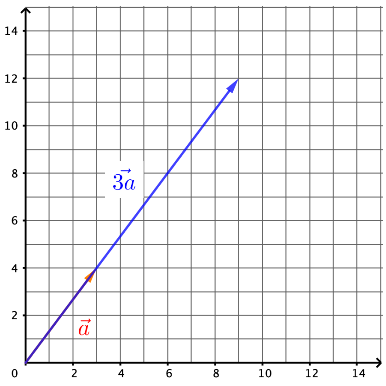
3) $2\vec{s} - \vec{t} + 2\vec{r}$

Scaled Vectors

Vectors $\vec{a}$ and $\overrightarrow{3a}$ are ________________.

$\vec{a} = \langle 3,4 \rangle \quad \overrightarrow{3a} = \langle 9,12 \rangle$

Calculate the magnitudes of $\vec{a}$ and $\overrightarrow{3a}$.
Show your work.



Is $|\overrightarrow{3a}|$ three times $|\vec{a}|$?

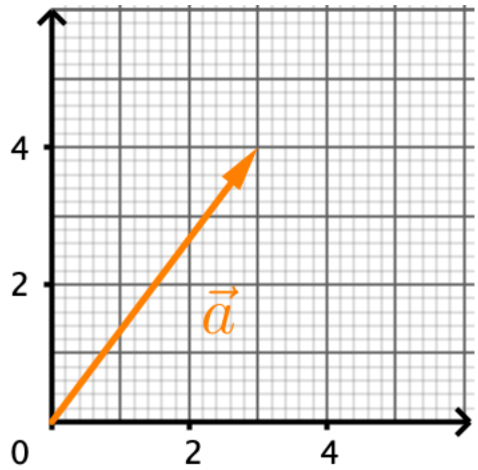
Unit Vectors

A unit vector is a vector with a magnitude of _________. We can turn any vector into a unit vector by multiplying each component be the _________________ of its magnitude.

$|\vec{a}| = 5 \qquad \vec{u} = \frac{1}{5}\langle 3,4 \rangle =$

Show $\vec{u}$ graphically on the grid to the right.

Does $\vec{u}$ have a magnitude of 1?
Justify your answer.



Linear Combination Form

Linear combination form expresses vectors as the sum of scaled vectors $\vec{\imath}$ and $\vec{\jmath}$, where $\vec{\imath}$ is a unit vector parallel to the ______________ and $\vec{\jmath}$ is a unit vector parallel to the ______________.

Write the component form of $\vec{\imath}$ and $\vec{\jmath}$. $\quad \vec{\imath} = \qquad\qquad \vec{\jmath} =$

Write $\vec{a}$ in component form and linear combination form. $\quad \vec{a} = \langle \quad , \quad \rangle =$ ___$\vec{\imath}$ + ___$\vec{\jmath}$

What should we take away?

- We can use vector operations such as addition, subtraction, and scalar multiplication to produce new vectors.
- Multiplying a vector by the reciprocal of its magnitude will produce a ________ vector going in the same direction.
- Vector $\langle a, b \rangle$ can also be expressed in linear combination form as __________________.

Topic 4.8 Vectors (Daily Video 3)

AP Precalculus

In this video, we will learn about the dot product and how to find the angle between vectors, as well as use the Law of Sines and Cosines to find side lengths and angle measures of triangles.

What should I remember ?

A vector is a quantity that has both _______________ and _________________.

> **Magnitude:** Given vector: $\vec{v} = \langle a, b \rangle$
>
> $$|\vec{v}| = \sqrt{a^2 + b^2}$$

The operations vector addition, vector subtraction, and scalar multiplication produce new vectors.

The Dot Product

> **Dot Product:** Given vector $\overrightarrow{v_1} = \langle a_1, b_1 \rangle$ and $\overrightarrow{v_2} = \langle a_2, b_2 \rangle$
>
> $$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = a_1 \cdot a_2 + b_1 \cdot b_2$$

Example: Evaluate $\vec{m} \cdot \vec{n}$, given $\vec{m} = \langle 3,5 \rangle$ and $\vec{n} = \langle -2,0 \rangle$. Show your work.

Note: The result of a dot product is a _____________ value, NOT a new vector.

> **Angle Between Vectors:** The angle between vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ can be found using:
>
> $$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = |\overrightarrow{v_1}||\overrightarrow{v_2}| \cos \theta$$

Let's look at an EXAMPLE!

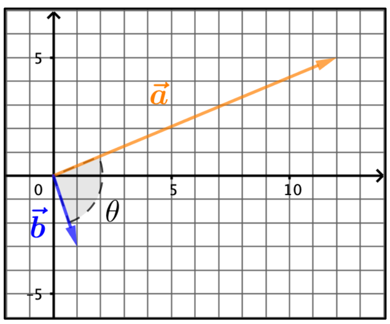
Find the angle between vectors $\vec{a} = \langle 12,5 \rangle$ and $\vec{b} = \langle 1, -3 \rangle$.

Begin by making the following calculations. Show your work.

$|\vec{a}| =$ $\qquad\qquad$ $|\vec{b}| =$ $\qquad\qquad$ $\vec{a} \cdot \vec{b} =$

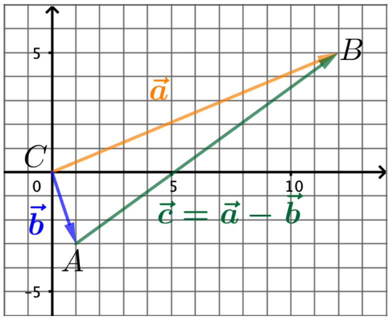
Now let's find the angle between the vectors $\vec{a}$ and $\vec{b}$. Show your work.

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$



Note: The graphs on the video are incorrect but all the calculations are correct.

Law of Sines and Cosines



Verify that $\vec{a}-\vec{b}$ has the same components as vector $\vec{c}=\langle 11,8\rangle$.

Calculate $|\vec{c}|$. Show your work.

Law of Cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Law of Sines:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Let's look at an EXAMPLE!

What are the lengths of the three sides? $a=|\vec{a}|=$ $\qquad$ $b=|\vec{b}|=$ $\qquad$ $c=|\vec{c}|=$

Solve for angle C. Show all work. Calculator tip: Store the value of C.	Solve for angle B. Show all work. Calculator tip: Use your stored value of C and store the value of B. Solve for angle A using your stored values of C and B.

What should we take away?

- The dot product is a vector operation that gives us a __________ quantity.
- The dot product can be used to find the ___________ between vectors.
- You can use the Law of Sine and Law of Cosines to solve for unknown ________ lengths and ____________ measures in a triangle.

Topic 4.9 Vector-Valued Functions (Daily Video 1)

AP Precalculus

In this video, we will look at how we can represent planar motion with vector-valued functions.

What have we already learned in Unit 4?

Topic 4.1: Parametric functions allow us to define two dependent variables from one independent variable, the parameter.

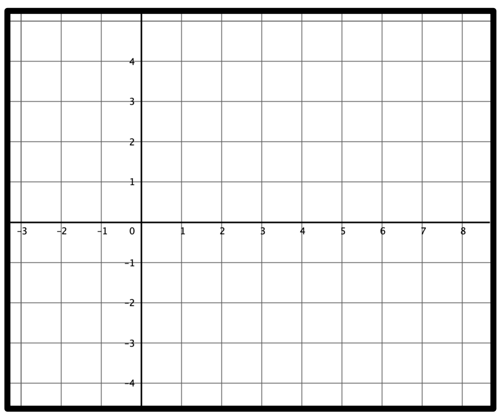
Topic 4.2: Parametric equations can be used to model the position of a particle at a given time.

Topic 4.8: Vectors are quantities that have both magnitude and direction.

Let's look at an EXAMPLE!

A particle moves along a curve such that its **position** at time t is given by $f(t) = \langle t^2 - 1, 1 - t \rangle$ where $-1 \leq t \leq 3$. Complete the table and then graph vectors on the grid below.

t	$x(t)$	$y(t)$	Vector
-1	0	2	
0	-1	1	
1	0	0	
2	3	-1	
3	8	-2	



Connect the heads of the vectors to see the path of the particle.

How far is the particle from the origin at $t = 2$? Show all work.

Velocity of a Particle

The vector-valued function $v(t) = \langle x(t), y(t) \rangle$ can be used to express the **velocity** of a particle moving in a plane at different times, t.

At time t, the sign of $x(t)$ indicates if the particle is moving __________ or ___________.

If $x(t) > 0$ then the particle is moving ________ and if $x(t) < 0$ then the particle is moving ________.

At time t, the sign of $y(t)$ indicates if the particle is moving __________ or ___________.

If $y(t) > 0$ then the particle is moving ________ and if $y(t) < 0$ then the particle is moving ________.

The magnitude of the velocity at time t gives the _____________ of the particle.

Let's PRACTICE!

At time t, the velocity of a particle moving in the xy-plane is given by the vector-valued function $v(t) = \langle x(t), y(t) \rangle$, where $x(t) = 2^t$ and $y(t) = t^2 - 6$.

a) What is the speed of the particle at $t = 3$? Show all work.

b) At time $t = 1$ is the particle moving left or right? Justify your response.

c) At time $t = 1$ is the particle moving up or down? Justify your response.

What should we take away?

If the **position** of a particle is modeled by the vector-valued function $p(t) = \langle x(t), y(t) \rangle$, then the magnitude of $p(t)$at time t gives the ______________ of the particle from the ____________.

If the **velocity** of a particle is modeled by the vector-valued function $v(t) = \langle x(t), y(t) \rangle$...

- The magitude of the velocity vector at time t gives the ____________ of the particle.
- At time t, the sign of ________ indicates if the particle is moving left or right.
- At time t, the sign of ________ indicates if the particle is moving up or down.

Topic 4.10 Matrices (Daily Video 1)

AP Precalculus

In this video, we will discuss what a matrix is, how it is used in the real world, and look at the components of a matrix.

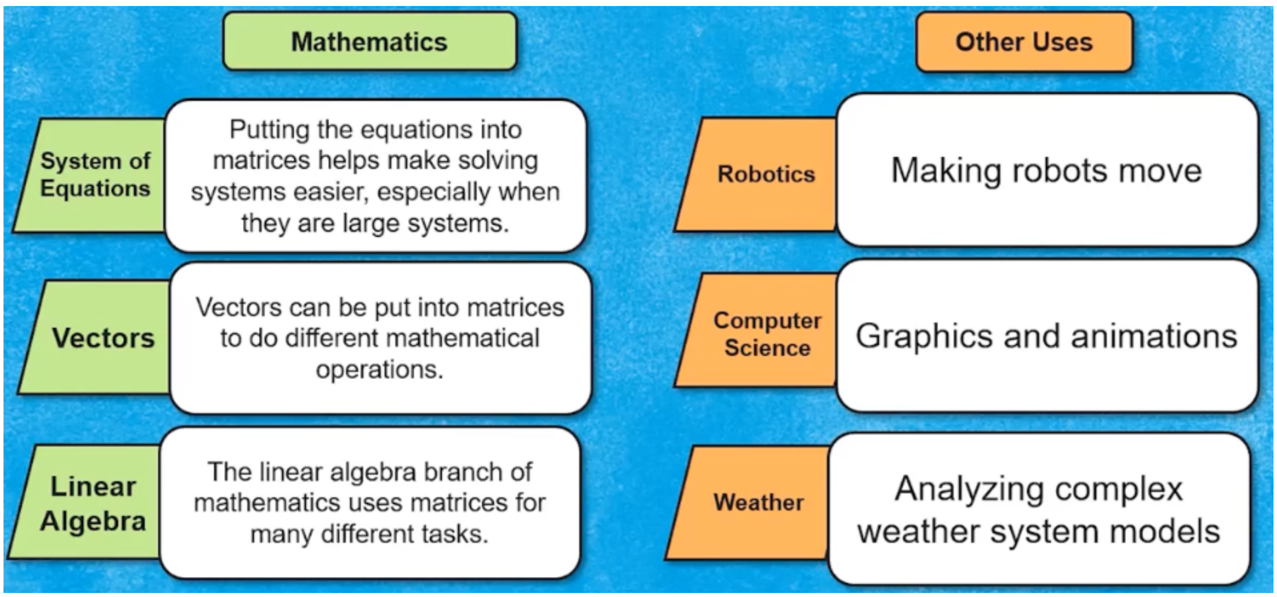
Downloading Songs

A **matrix** is a rectangular array of numbers organized in __________ and ___________.

Here are our data on the different genres of songs put into a matrix:

	Pop	Country	Hip Hop
Week 1	12,243	10,982	13,239
Week 2	13,720	15,673	15,951
Week 3	10,239	16,783	21,182
Week 4	15,453	11,298	28,901

When do we use matrices?



Vectors as a Matrix

A vector can be written as either a **row matrix** or a **column matrix:**

A row matrix is just one row: $\begin{bmatrix} 3 & 5 \end{bmatrix}$ A column matrix is just one column: $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Example: Write the vector $2\vec{\imath} + 8\vec{\jmath}$ as a column matrix. ____________________

How do you name a matrix?

- ☐ A matrix is named with a bold capital letter
- ☐ The **order** of a matrix is also known as the ___________ is written as **rows** × **columns**.

Examples: What is the order of the following matrices?

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 1 & 9 \\ 1 & 4 & 28 & 11 \end{bmatrix}$$

Order:___________

$$\mathbf{B} = \begin{bmatrix} 0 & 4 & -8 \\ -1 & 28 & 11 \\ 2 & 11 & 5 \\ 0 & 2 & 4 \end{bmatrix}$$

Order:___________

Let's PRACTICE!

What is the order of the matrix?

1. $\begin{bmatrix} -1 & -12 \\ 3 & 2 \\ 5 & 6 \end{bmatrix}$

2. $\begin{bmatrix} 2 & 3 & -9 & 15 \end{bmatrix}$

3. $\begin{bmatrix} 2 & 4 & 1 & -10 & 0 & 1 \\ 5 & 7 & 9 & 3 & 2 & 1 \end{bmatrix}$

Elements of a Matrix

An **element** of a matrix **A** is an entry in the matrix. It would look like a_{ij}, where i is the row and j is the column where the element is located in matrix **A**.

For the matrix $\mathbf{A} = \begin{bmatrix} -3 & 6 \\ 10 & 0 \\ 12 & 5 \end{bmatrix}$, what entry is a_{21}? _________

Let's PRACTICE!

Given the matrix $\mathbf{B} = \begin{bmatrix} 2 & 4 & 1 & -10 & 0 & 1 \\ 5 & 7 & 9 & 3 & 2 & 1 \end{bmatrix}$, identify the following elements:

1. $b_{12} =$

2. $b_{25} =$

3. $b_{22} =$

Let's REVIEW!

$$\mathbf{M} = \begin{bmatrix} 12{,}243 & 10{,}982 & 13{,}239 \\ 13{,}720 & 15{,}673 & 15{,}951 \\ 10{,}239 & 16{,}783 & 21{,}182 \\ 15{,}453 & 11{,}298 & 28{,}901 \end{bmatrix}$$

a) What is the order of $\mathbf{M}$? _____________ b) What is element m_{23}? ______________

What should we take away?

- ☐ A **matrix** is a rectangular array of numbers organized in __________ and ___________.
- ☐ Matrices have many different uses in mathematics and the broader STEM world.

Topic 4.10 Matrices (Daily Video 2)

AP Precalculus

In this video, we will discuss how to multiply two matrices.

Multiplying Matrices Two matrices can be multiplied together if the number of **columns** in the first matrix equals the number of **rows** in the second matrix.

Example: Can the two matrices below be multiplied?

$\begin{bmatrix} 3 & 1 \\ 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & 8 \\ 6 & 2 & 1 \end{bmatrix}$ The matrices $\underbrace{\hspace{2cm}}_{\text{can;can not}}$ be multiplied. Give a reason for your answer.

If you can multiply these matrices what order is the answer matrix?

Can these matrices be multiplied?

State whether these matrices can be multiplied. If they can, what is the order of the resultant product matrix?

1. $\begin{bmatrix} 6 & 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 8 \\ 4 \end{bmatrix}$	2. $\begin{bmatrix} 1 & 5 & -2 \\ -11 & 3 & -9 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$	3. $\begin{bmatrix} -2 & 0 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} -1 & 4 \\ 0 & 1 \\ 2 & 7 \end{bmatrix}$

Multiplying Matrices: Dot Product

Dot Product for Matrices: $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} =$ ______________________

Find the dot product. Show your work.

$\begin{bmatrix} -1 & 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix} =$

Multiplying Matrices: Example

In **matrix multiplication**, the product of two matrices is a new matrix in which the element in the i^{th} row and j^{th} column is the dot product of the i^{th} row of the first matrix and the j^{th} column of the second matrix.

Multiply and show all your work.

$\begin{bmatrix} 1 & 5 & -2 \\ -11 & 3 & -9 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 4 & 3 \\ -4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 0 & -4 & 0 \\ -1 & 8 & 6 \end{bmatrix}$

Concert Tickets

A concert venue sold tickets to a three-day music festival. The rows of the first matrix show the tickets sold for each day, and the columns show the number of tickets sold for the different types of seats. If covered seats cost $50 and lawn seats cost $25, how much money did the venue make each day? Show your work.

Explain why this matrix product gives how much money the venue made each day.

$$\begin{array}{r} \\ \text{Friday} \\ \text{Saturday} \\ \text{Sunday} \end{array} \begin{array}{c} \begin{array}{cc} \text{Covered Seats} & \text{Lawn Seats} \end{array} \\ \begin{bmatrix} 1{,}200 & 2{,}200 \\ 1{,}450 & 3{,}000 \\ 1{,}375 & 2{,}600 \end{bmatrix} \end{array} \cdot \begin{bmatrix} 50 \\ 25 \end{bmatrix} \begin{array}{l} \text{cost per covered seat} \\ \text{cost per lawn seat} \end{array}$$

Calculate: $\begin{bmatrix} 1{,}200 & 2{,}200 \\ 1{,}450 & 3{,}000 \\ 1{,}375 & 2{,}600 \end{bmatrix} \cdot \begin{bmatrix} 50 \\ 25 \end{bmatrix}$

What should we take away?

To multiply matrices, the number of ________________in the first matrix must be equal to the number of ________________in the second matrix.

Then, find the _______ product of the i^{th} row of the first matrix and the j^{th} column of the second matrix and continue this process until the multiplication is complete.

Note: Matrix multiplication is **not** commutative

Topic 4.11 The Inverse and Determinant of a Matrix (Daily Video 1)

AP Precalculus

In this video, we will discuss how to find the inverse of a 2 x 2 matrix and what applications this has in mathematics.

What is a mathematical identity?

What number is the additive identity? _____ What number is the multiplicative identity? _____

What is special about an identity?

Identity Matrix: An **identity matrix** is a square matrix (equal numbers of rows and columns) that contains ones (1) along the main diagonal starting in the top left and going to the bottom right, with all other elements being zeros. $\boldsymbol{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\boldsymbol{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Multiply and show your work.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 6 \\ 4 & -2 \end{bmatrix} =$ $\begin{bmatrix} -3 & 6 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$

Multiply and show your work. What do you notice about the product of these two matrices?

$\begin{bmatrix} -6 & 3 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{12} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} \end{bmatrix} =$

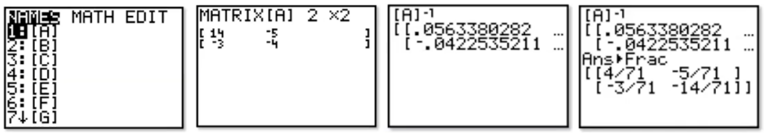
Inverse Matrix

Given matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse matrix $\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$

Example A	Example B
Given matrix $\mathbf{A} = \begin{bmatrix} 8 & -2 \\ 6 & 5 \end{bmatrix}$, find the inverse matrix. Show your work. Verify that $\mathbf{A} \cdot \mathbf{A}^{-1} = \boldsymbol{I_2}$	Given matrix $\mathbf{B} = \begin{bmatrix} -4 & 2 \\ 12 & 10 \end{bmatrix}$, the find inverse matrix. Show your work. Verify that $\mathbf{B} \cdot \mathbf{B}^{-1} = \boldsymbol{I_2}$

Finding the Inverse with a Calculator *If you used a different calculator, ask your teacher.

To find the inverse on a graphing calculator, first input the matrix.



What should we take away?

- ☐ An identity matrix, I, is a square matrix that contains ones along the main diagonal starting in the top left and going to the bottom right, with all other elements being zeros.
- ☐ Multiplying a square matrix by its corresponding identity matrix results in the ____________ matrix.
- ☐ The product of a square matrix and its inverse is the ____________ matrix.
- ☐ The inverse of a 2 x 2 matrix can be calculated with or without technology.

Topic 4.11 The Inverse and Determinant of a Matrix (Daily Video 2)

AP Precalculus

In this video, we will discuss how to find the determinant* of a 2 x 2 matrix and how this can be used to find the area of a parallelogram spanned by two vectors.

*video text shows **inverse** but presenter says **determinant.**

What is a determinant? $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det(A) = ad - bc$

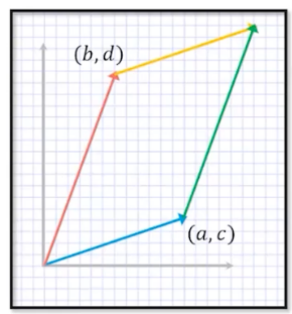
What can determinants be used for?

- ☐ To calculate the inverse martix $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- ☐ Find the area of certain geometric figures, such as triangles and parallelograms.
- ☐ To know if a system of equations has a unique solution.
- ☐ To see how linear transfomations change.

Let's look at an EXAMPLE!

Find the determinant of $\mathbf{A} = \begin{bmatrix} -16 & -12 \\ 4 & -1 \end{bmatrix}$. Show your work.	Find the determinant using a calculator. **Edit** **Math** 

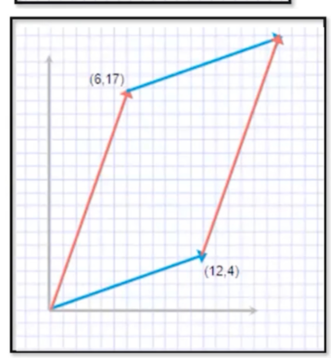
Let's PRACTICE!



Using the determinant to find the area of a parallelogram formed by four vectors. We can fully define a parallelogram using only two vectors and those vectors can be represented as a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Area $= |\det(A)|$ If $\det(A) = 0$ then the vectors are ____________.

Find the area of the parallelogram to the right.



Write the two vectors in the parallelogram as column matrices.

$\begin{bmatrix} \quad \\ \quad \end{bmatrix}$ and $\begin{bmatrix} \quad \\ \quad \end{bmatrix}$. Make them into one 2×2 matrix $\mathbf{A} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

Calculate the area of the parallelogram. Show your work.

You can count the squares to verify your answer is reasonable.

What should we take away?

- ☐ The determinant of matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) =$ ______________.
- ☐ The area of a parallelogram spanned by vectors is Area = ______________.
- ☐ If $\det(A) = 0$ then the vectors are ____________.

Topic 4.12 Linear Transformations and Matrices (Daily Video 1)

AP Precalculus

In this video, we will learn how to perform a linear transformation using a 2×2 matrix.

Let's REVIEW!

Review of Topic 4.10 :

Calculate the product of the two matrices.

$\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} =$

Functions:

Algebra: $f(x) = 2x^2 + 3$

If we have an input of $x = 3$, the output is

$y = f(3) =$ ______________

Let's look at an EXAMPLE!

Linear Transformation: Vocabulary and Notation

Input Vector: $\vec{v}$

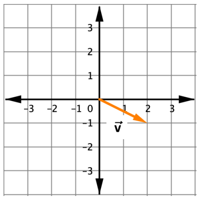
Linear Transformation: $L(\vec{v})$

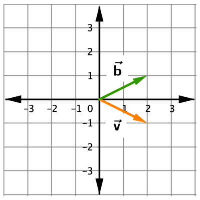
Output Vector (a new vector): $\vec{b}$

A is a unique 2×2 matrix.

$L(\vec{v}) = \mathbf{A}\vec{v} = \vec{b}$

Linear Transformation Example





$\vec{v} = \langle 2, -1 \rangle \qquad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

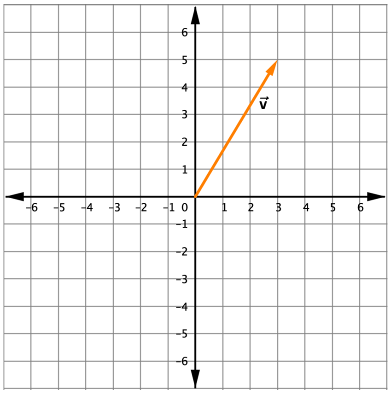
Calculate $\vec{b}$. Show your work.

$L(\vec{v}) = \mathbf{A}\vec{v} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \vec{b}$

$\vec{b} =$

Describe how $L(\vec{v})$ changes $\vec{v}$.

Let's PRACTICE!



Given $\vec{\mathbf{v}} = \langle 3,5 \rangle$ and A $= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, perform and describe the linear transformation $L(\vec{v}) = \mathbf{A}\vec{v} = \vec{b}$.

Perform the linear transformation. Show your work.

Graph the output vector $\vec{b}$ on the grid.

Describe how $L(\vec{v})$ changes $\vec{v}$.

What should we take away?

- ☐ A linear transformation maps an input vector to an output vector.
- ☐ A linear transformation is a function that involves a vector and a 2×2 matrix.

Topic 4.13 Matrices as Functions (Daily Video 1)

AP Precalculus

In this video, we will learn how to determine the association between a linear transformation and a matrix.

Let's WARM UP!

Review of Topic 4.12 :

Given $\vec{v} = \langle 3, 5 \rangle$ and A $= \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$, perform the linear transformation to obtain an output vector.

$$A\vec{v} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} \underline{(\ \ )(\ \ )} + \underline{(\ \ )(\ \ )} \\ \underline{(\ \ )(\ \ )} + \underline{(\ \ )(\ \ )} \end{bmatrix} = \begin{bmatrix} 14 \\ 6 \end{bmatrix}$$

The output vector is $\langle$_____,____$\rangle$.

Review of Topic 4.11:

Remember: Given matrix A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $ad - bc$.

Practice: Given A $= \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}$, find the determinant.

$$\det(A) = \underline{(\ \ )(\ \ )} - \underline{(\ \ )(\ \ )} = \_\_\_\_$$

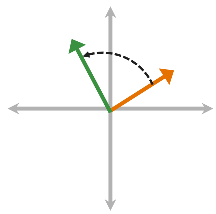
Let's look at an EXAMPLE!

Counterclockwise Rotation About the Origin

Given $\vec{v}$ and A $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, the linear transformation $A\vec{v}$ will map the new vector in a counterclockwise rotation of an angle, θ, about the origin.

If A $= \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix}$, then input vector is mapped 90 degrees counterclockwise.

Notice that A $= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which also rotates the input vector 90 degrees counterclockwise.



Let's PRACTICE!

For a linear transformation, $A\vec{v}$, which matrix will rotate a given vector 270 degrees counterclockwise about the origin?

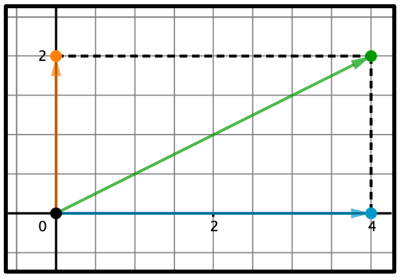
(A) A $= \begin{bmatrix} \cos\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \end{bmatrix}$

(B) A $= \begin{bmatrix} \cos\frac{3\pi}{2} & -\sin\frac{3\pi}{2} \\ \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2} \end{bmatrix}$

(C) A $= \begin{bmatrix} \sin\frac{\pi}{2} & \sin\frac{\pi}{2} \\ -\cos\frac{\pi}{2} & -\cos\frac{\pi}{2} \end{bmatrix}$

(D) A $= \begin{bmatrix} \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2} \\ -\cos\frac{3\pi}{2} & \sin\frac{3\pi}{2} \end{bmatrix}$

Let's look at an EXAMPLE! Determinants and Transformations



Given a rectangle defined by these four vectors:

$\vec{a} = \langle 0, 0 \rangle$ $\quad \vec{b} = \langle 4, 0 \rangle$ $\quad \vec{c} = \langle 0, 2 \rangle$ $\quad \vec{d} = \langle 4, 2 \rangle$

What is the area of this rectangle *Rec*? Show your work.

We will use a transformation matrix, M $= \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$, to find the four output vectors.

$\vec{a} = \langle 0,\ 0 \rangle$ $\quad \vec{b} = \langle 4,\ 0 \rangle$ $\quad \vec{c} = \langle 0,\ 2 \rangle$ $\quad \vec{d} = \langle 4,\ 2 \rangle$

Find the four output matrices. Show your work.

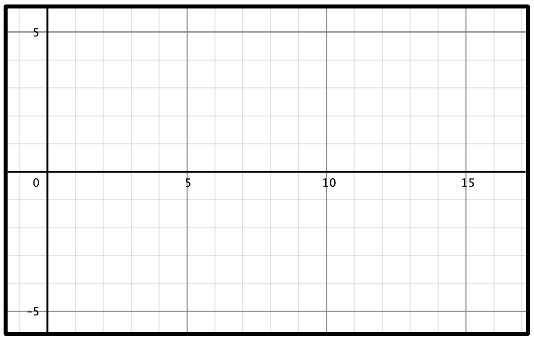
$M\vec{a} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} =$

$M\vec{b} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix} =$

$M\vec{c} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} =$

$M\vec{d} = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} =$

Show the four output matrices on the grid to the right and show the quadrilateral that is the result of this linear transformation.



Calculate $\det(M)$. Show your work.

The absolute value of the determinant of the transformation matrix is the scaling factor for the area of the transformed region.

The area of our new quadrilateral is ___________ times the original area 8. What is the area of the transformed quadrilateral? __________

Use the grid above to verify that the area of the transformed quadrilateral has an area of 40. Show all work.

Let's PRACTICE! In a linear transformation, a region with area $6\ \text{cm}^2$ is mapped onto a region with area $60\ \text{cm}^2$. Given the transformation matrix A $= \begin{bmatrix} x & 5 \\ x+1 & 8 \end{bmatrix}$, find the positive value of x. Show your work.

A. $x = 5/3$ B. $x = 3$

C. $x = 5$ D. $x = 10$

What should we take away?

- ☐ We can write a transformation matrix that will rotate a vector counterclockwise about the origin.
- ☐ Starting with a region defined by vectors, we can use a transformation matrix to create a transformed region.
- ☐ We can use the absolute value of the determinant of a transformation matrix to compute the area of the transformed region.
- ☐ The absolute value of the determinant of a transformation matrix is the scaling factor, or the magnitude of the dilation, of the region.

Topic 4.13 Matrices as Functions (Daily Video 2)

AP Precalculus

In this video, we will learn how to determine the composition of two linear transformations.

Let's REVIEW!

Composition of Functions (Topic 2.7) Given $f(x) = 2x + 3$ and $g(x) = x^2 - 4$, find $f\big(g(3)\big)$.	
Method 1 Find $g(3) =$ ______________ $= 5$ Then find $f(5) =$ ______________ $= 13$	Method 2 Find $f\big(g(x)\big) =$ ____________________ Then find $f\big(g(3)\big) =$ ______________ $= 13$

Remember: the composition of functions is <u>not</u> commutative

Note: Show your work means **pause** the video then check your work with the video.

Multiplying Matrices (Topic 4.10) Find the product of the two matrices below. Show your work. $\begin{bmatrix} 1 & -2 \\ 7 & 4 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}$

Let's look at an EXAMPLE!

Chaining Transformations Method Given $\vec{\boldsymbol{u}} = \langle -2, 3\rangle$, $\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, find the output vector $\vec{\boldsymbol{z}} = B(A\vec{\boldsymbol{u}})$.	
Transformation 1 First, find the linear transformation $A\vec{\boldsymbol{u}} = \vec{\boldsymbol{v}}$. Show your work. $A\vec{\boldsymbol{u}} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \quad \end{bmatrix} =$	Transformation 2 Now find the linear transformation $B\vec{\boldsymbol{v}} = \vec{\boldsymbol{z}}$. Show your work. $B\vec{\boldsymbol{v}} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} \quad \end{bmatrix} =$
Multiplying the Matrices Method Find the product of the linear transformation matrices first. Show your work. $BA = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} =$	Now find the product $B(A\vec{\boldsymbol{u}})$. Show your work. $B(A\vec{\boldsymbol{u}}) = \begin{bmatrix} 2 & -2 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} \quad \end{bmatrix} =$

Let's PRACTICE!

Given $\vec{\boldsymbol{v}} = \langle 1, -2 \rangle$, $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$, find the matrix $\vec{\mathbf{z}}$, where $\vec{\mathbf{z}} = B(A\vec{\boldsymbol{v}})$. Show your work.

A $\vec{\mathbf{z}} = \langle 7,\ 16 \rangle$

B $\vec{\mathbf{z}} = \langle -5,\ 4 \rangle$

C $\vec{\mathbf{z}} = \langle 1,\ 4 \rangle$

D $\vec{\mathbf{z}} = \langle 7,\ -2 \rangle$

What should we take away?

The composition of two linear transformations is similar to the composition of functions.

We can determine the composition of two linear transformations in two ways:

- We can chain the linear transformations to find the output vector.
- We can multiply the matrices associated with the composition and then determine the output vector.

Topic 4.13 Matrices as Functions (Daily Video 3)

AP Precalculus

In this video, we will learn how to determine the inverse of a linear transformation.

Let's WARM UP!

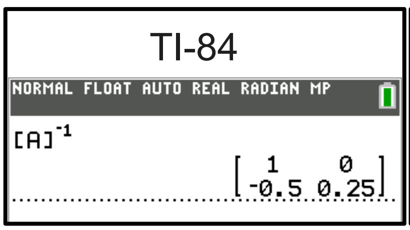
Multiplying a Matrix and Its Inverse (Topic 4.11) Multiply matrix **A** and matrix **B**. $\mathbf{A}=\begin{bmatrix}1 & -2\\0 & 2\end{bmatrix}$ $\mathbf{B}=\begin{bmatrix}1 & 1\\0 & \frac{1}{2}\end{bmatrix}$	
Find **AB**. Show your work.	Find **BA**. Show your work.

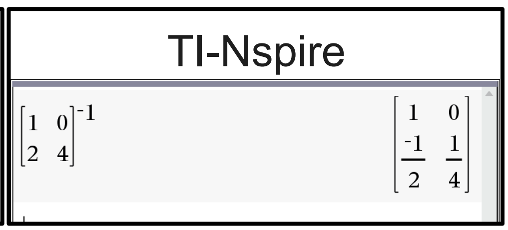
Both products produce an identity matrix, which indicates that matrix A and matrix B are inverses of each other.

Let's REVIEW!

Determine the Inverse of a 2×2 Matrix (Topic 4.11)

Use technology to find the inverse of matrix $\mathbf{A}$, which is $\mathbf{A}^{-1}$. $\mathbf{A}=\begin{bmatrix}1 & 0\\2 & 4\end{bmatrix}$





$$\mathbf{A}^{-1}=\begin{bmatrix}1 & 0\\-\frac{1}{2} & \frac{1}{4}\end{bmatrix}$$

Let's look at an EXAMPLE!

Compositions of Linear Transformations Given $\vec{\boldsymbol{u}}=\langle 3,-4\rangle$, $\mathbf{A}=\begin{bmatrix}0 & 3\\1 & 2\end{bmatrix}$, and $\mathbf{B}=\begin{bmatrix}-\frac{2}{3} & 1\\\frac{1}{3} & 0\end{bmatrix}$	
Find the output vector for the composite $\mathbf{A}(\mathbf{B}\vec{\boldsymbol{u}})$. Show your work. $\begin{bmatrix}0 & 3\\1 & 2\end{bmatrix}\begin{bmatrix}-\frac{2}{3} & 1\\\frac{1}{3} & 0\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix} \quad \end{bmatrix}$	Find the output vector for the composite $\mathbf{B}(\mathbf{A}\vec{\boldsymbol{u}})$. Show your work. $\begin{bmatrix}-\frac{2}{3} & 1\\\frac{1}{3} & 0\end{bmatrix}\begin{bmatrix}0 & 3\\1 & 2\end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix} \quad \end{bmatrix}$

These two linear transformations are inverses, because their composition maps the given vector onto itself.

Finding an Inverse Transformation

A linear transfomation, L, is given by $L(\vec{\boldsymbol{v}}) = \mathbf{A}\vec{\boldsymbol{v}} = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$. Find its inverse transformation, $L^{-1}(\vec{\boldsymbol{v}})$. Use technology to find the inverse matrix $\mathbf{A}^{-1}$.

$\mathbf{A}^{-1} = \begin{bmatrix} \quad \end{bmatrix}$ Find $L^{-1}(\vec{\boldsymbol{v}}) = \mathbf{A}^{-1}\vec{\boldsymbol{v}}$ without technology. Show your work.

The output vector $\mathbf{A}^{-1}\vec{\boldsymbol{v}} =$ ______________

Let's check our work without technology. Show your work.

$\mathbf{A}(\mathbf{A}^{-1}\vec{\boldsymbol{v}}) = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \frac{7}{2} \\ -16 \end{bmatrix} =$

$\mathbf{A}^{-1}(\mathbf{A}\vec{\boldsymbol{v}}) = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ -4 \end{bmatrix} =$

Let's PRACTICE!

A linear transfomation, L, is given by $L(\vec{\boldsymbol{v}}) = \mathbf{A}\vec{\boldsymbol{v}}$.

Given $\vec{\boldsymbol{v}} = \langle 5, -1 \rangle$, $\boldsymbol{A} = \begin{bmatrix} 3 & 6 \\ 0 & 1 \end{bmatrix}$, find its inverse transfomation, $L^{-1}(\vec{\boldsymbol{v}})$. Use technology to find $\mathbf{A}^{-1}$ and $\mathbf{A}^{-1}\vec{\boldsymbol{v}}$ without technology. Show your work.

A $\left\langle \frac{3}{2}, -\frac{1}{4} \right\rangle$

B $\langle 9, -4 \rangle$

C $\left\langle -\frac{31}{3}, 5 \right\rangle$

D $\left\langle \frac{11}{3}, -1 \right\rangle$

$L^{-1}(\vec{\boldsymbol{v}}) = \mathbf{A}^{-1}\vec{\boldsymbol{v}} = \begin{bmatrix} \quad \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix} =$

What should we take away?

- Two linear transformations are inverses if their composition maps a vector to itself.
- To determine the inverse of a linear transformation, find the inverse transformation by using the inverse of the given matrix.

Topic 4.14 Matrices Modeling Contexts (Daily Video 1)

AP Precalculus

In this video, we will learn how to construct a model involving transitions using matrices.

Let's WARMUP!

Probability

To get to school, you have the option to ride the school bus or ride your bike. The probability that you ride on the school bus is 65%. What is the probability that you ride your bike?

Let's look at an EXAMPLE!

Transition Probability Matrix

A **transition probability matrix** is a square matrix. We will only work with 2×2 matrices in our examples and problems.

$$\begin{matrix} & A & A' \\ A & r & s \\ A' & 1-r & 1-s \end{matrix}$$

r is the probability of starting in state A and ending in state A.

To get to school, you have the option to ride the school bus or ride your bike. If you ride the bus one day, the probability you ride the bus the next day is 70%.

s is the probability of starting in state A' and ending in state A. If you ride your bike one day, the probability you will ride the bus the next day is 25%.

$$\begin{matrix} & bus & bike \\ bus & 0.7 & 0.25 \\ bike & \_\_\_\_ & \_\_\_\_ \end{matrix}$$

The transition probability matrix tells us that:

- $P(\text{bus today, bus tomorrow}) = 0.7$
- $P(\text{bus today, bike tomorrow}) = 0.3$
- $P(\text{bike today, bus tomorrow}) = 0.25$
- $P(\text{bike today, bike tomorrow}) = 0.75$

If you rode the bus on Monday, what is the probability that you will ride the bus on Wednesday?

$$T = \begin{matrix} & bus & bike \\ bus & 0.70 & 0.25 \\ bike & 0.30 & 0.75 \end{matrix}$$

We need to compute $T \cdot T = T^2 = \begin{bmatrix} 0.70 & 0.25 \\ 0.30 & 0.75 \end{bmatrix}^2$. Use technology to compute $T^2 = \begin{bmatrix} & \\ & \end{bmatrix}$.

The probability that you will ride the bus on Wednesday given that you rode the bus on Monday is ________________.

Initial State Vector

The **initial state vector** is a column matrix. It has only one column. There is a row for each state.

$A = \text{State } A$ $\quad A' = \textbf{Not} \text{ state } A$ $\quad v_0 = \begin{bmatrix} A \\ A' \end{bmatrix}$

Example: UPS has 88 drivers for a city. The city is divided into a north region and a south region. Each day, drivers are assigned to deliver to one of these two regions. Today, 48 of the drivers were assigned to the north region. Write an initial state vector.

$$v_0 = \begin{bmatrix} North \\ South \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Example: To get to school, you have the option to ride the school bus or ride your bike. The probability that you ride on the school bus is 65%.

$$v_0 = \begin{bmatrix} B \\ B' \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Let's PRACTICE!

A food truck offers a choice between a hamburger and a hotdog. Of customers who order a hamburger on their first visit, 60% will order a hamburger on their second visit. Of customers who order a hotdog on their first visit, 55% will order a hotdog on their second visit. Write a transition probability matrix to model this scenario.

Use the labels B for hamburger and D for hotdog.

$$\begin{array}{c c} & \begin{array}{cc} B & D \end{array} \\ \begin{array}{c} B \\ D \end{array} & \begin{bmatrix} 0.60 & \square \\ \square & \square \end{bmatrix} \end{array}$$

What should we take away?

- We can write a transition probability matrix.
- We can determine probabilities using a transition probability matrix.
- We can use a transition probability matrix over discrete intervals to model how states change.
- We can write an initial state vector (or column matrix).

Topic 4.14 Matrices Modeling Contexts (Daily Video 2)

AP Precalculus

In this video, we will apply matrix models to predict future and past states.

Let's WARMUP!

Writing an Initial State Vector

A small town has two movie theaters, GoShow and Cinema1. Seventy-five people go to the movies one night. Fifty people go to the GoShow theater. Write an initial state vector.

$$v_0 = \begin{matrix} \text{GS} \\ \text{C1} \end{matrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Let's REVIEW!

A small town has two movie theaters, GoShow and Cinema1. Of customers who go to the GoShow on one visit, 62% will return to the GoShow for their next visit. Of customers who go to the Cinema1 on one visit, 55% will return to the Cinema1 on their next visit.

Write a transition probability matrix to model this scenario. Use the labels GS for GoShow and $C1$ for Cinema1.

	GS	C1
GS	0.62	☐
C1	☐	☐

Let's look at an EXAMPLE!

Predicting Future States

A small town has two movie theaters, GoShow and Cinema1.

$$v_0 = \begin{matrix} GS \\ C1 \end{matrix} \begin{bmatrix} 50 \\ 25 \end{bmatrix} \qquad T = \begin{bmatrix} 0.62 & 0.45 \\ 0.38 & 0.55 \end{bmatrix}$$

To predict how many of the initial 75 customers go to each movie theater on their **second** visit, we need to multiply the matrix and the vector, Tv_0. Use technology to multiply the matrices.

$$Tv_0 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

On their second visit to a movie theater, about ________ of the original 75 people will go to the GoShow and about _________ will go to the Cinema1.

Predicting Future States

To predict how many customers go to each movie theater on their **third** visit, we need to find $T^2 v_0$.

$$T^2 v_0 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}^2 \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

On their third visit to a movie theater, about ________ of the original 75 people will go to the GoShow and about _________ will go to the Cinema1.

Predicting Past States

At the end of 2020, Smalltown had a population of 3200 and Tinyville had a population of 5000. It is known that people move between the two towns. Each year 3% of the population of Smalltown moves to Tinyville. Also 5% of the population of Tinyville moves to Smalltown.

$$v_0 = \begin{matrix} S \\ T \end{matrix} \begin{bmatrix} 3200 \\ 5000 \end{bmatrix} \qquad T = \begin{matrix} \\ S \\ T \end{matrix} \begin{matrix} \begin{matrix} S & \quad T \end{matrix} \\ \begin{bmatrix} \square & 0.05 \\ 0.03 & \square \end{bmatrix} \end{matrix}$$

In order to predict past states, we use inverse matrices.

At the end of 2019, what was the population of Smalltown and Tinyville? We need to calculate $T^{-1} v_0$. Use technology to find the matrix T^{-1} and calculate the value of $T^{-1} v_0$.

$$T^{-1} v_0 = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}^{-1} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

At the end of 2019, the population of Smalltown was about _________ and the population of Tinyville was about ___________________.

What should we take away?

- Using a transition matrix and an initial state vector, we can predict future and past states.
- Some questions will ask for probabilities.
- Some questions will ask for amounts.
- We can use technology to multiply matrices and to find the inverse of a matrix.

Topic 4.14 Matrices Modeling Contexts (Daily Video 3)

AP Precalculus

In this video, we will explore several contextual matrix models.

Let's WARMUP!

Transition Matrix

An insurance company classifies drivers as low-risk or not low-risk and tracks drivers' risk categories over time.

The company has found that 94% of drivers in the low-risk category one year will remain in that category the next year, and that 76% of drivers who are **not** in the low-risk category one year **will** be in the low-risk category the next year.

Write a transition matrix, T. Use L and L' for low-risk and not low-risk.

$$T = \begin{matrix} & L & L' \\ L & \square & \square \\ L' & \square & \square \end{matrix} \qquad v_0 = \begin{matrix} L & \square \\ L' & \square \end{matrix}$$

The insurance company studied a sample of 250 clients over several years. In the current year, 220 of these clients are low-risk. Write an initial state vector, v_0.

Let's PRACTICE!

Practice Question with Technology

An insurance company classifies drivers as low-risk or not low-risk and has found that 94% of drivers in the low-risk category one year will remain in that category the next year, and that 76% of drivers who are **not** in the low-risk category one year **will** be in the low-risk category the next year. The insurance company studied a sample of 250 clients over several years. In the current year, 220 of these clients are low-risk.

(A) What is the probability that a driver who is low-risk in the current year will still be a low-risk driver in two years?

$T = \begin{bmatrix} 0.94 & 0.76 \\ 0.06 & 0.24 \end{bmatrix}$ We need to find T^2. Use technology to calculate T^2.

$$T^2 = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

The probability that a current low-risk driver will still be a low-risk driver in two years is about ______________.

(B) How many clients who are low-risk drivers in the current year are predicted to **not** be low-risk in three years? We need to find $T^3\ v_0$.

$$T^3\ v_0 = \begin{bmatrix} 0.94 & 0.76 \\ 0.06 & 0.24 \end{bmatrix}^3 \begin{bmatrix} \square \\ \square \end{bmatrix} = \begin{bmatrix} \square \\ \square \end{bmatrix}$$

For a current low-risk driver, about ________ will not be low-risk in three years.

(C) How many clients who are low-risk drivers in the current year were not low-risk one year ago? We need to find $(T^{-1})v_0$.

$$(T^{-1})\ v_0 = \begin{bmatrix} 0.94 & 0.76 \\ 0.06 & 0.24 \end{bmatrix}^{-1} \begin{bmatrix} 220 \\ 30 \end{bmatrix} = \begin{bmatrix} \square \\ \square \end{bmatrix}$$

For a current low-risk driver, about ________ were not low-risk one year ago.

What should we take away?

- Starting with a contextual scenario, we can create initial state vectors and transition matrices to solve problems.
- We can use repeated multiplication of a matrix to model transitions between states to predict a future state.
- We can use repeated multiplication of the inverse of a matrix to model transitions between states to predict past states.
- We can use technology to multiply matrices and to find the inverse of a matrix.