Mathematical Practice 7: Look for and Make Use of Structure

Elementary School

<table>
<tr><td colspan="2">134-39</td></tr>
<tr><td>Non-structural thinking:</td><td>Structural thinking:</td></tr>
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$$\begin{array}{rccc} & & 12 & \\ & 0 & \cancel{2} & 14 \\ & \cancel{1} & \cancel{3} & \cancel{4} \\ - & & 3 & 9 \\ \hline & & 9 & 5 \end{array}$$

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- $134 - 34 - 5 = 95$ (Seeing the 39 as being composed of a 34 and a 5)
- $134 - 40 + 1 = 95$ (Subtract a nice number, then adjust)
- $135 - 40 = 95$ (Make an equivalent problem by understanding subtraction as a distance)

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A line passes through the points (2, 5) and (7, -4). What is the y-coordinate of the point on the line that has an x-coordinate of 12?

Non-structural thinking:

Find the slope: $\frac{-4-5}{7-2} = -\frac{9}{5}$

Solve for the y-intercept:

$$5 = -\frac{9}{5}(2) + b \Rightarrow b = 5 + \frac{18}{5} = \frac{25+18}{5} = \frac{43}{5}$$

Write an equation for the line:

$$y = -\frac{9}{5}x + \frac{43}{5}$$

Plug in $x = 12$.

$$y = -\frac{9}{5}(12) + \frac{43}{5} = -\frac{108}{5} + \frac{43}{5} = -\frac{65}{5} = -13$$

Structural thinking:

The given two points show that a change of 5 horizontal units corresponds with a change of -9 vertical units. An x-coordinate of 12 is another 5 units away from x=7, so the y-coordinate must be 9 units lower. $-4 - 9 = -13$.

The diagram below captures what students thinking structurally are able to "see" in their head.

+5 +5

x	2	7	12
y	5	-4	

-9 -9

Algebra 2

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Given the system of equations given below, find $2x - y$.

$$\begin{cases} 4x - 8y = 13 \\ 2x - 7y = -1 \end{cases}$$

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I have to first solve the system.
Multiply the bottom equation by 2.

$$4x - 8y = 13$$
$$2(2x - 7y) = 2(-1)$$

$$4x - 8y = 13$$
$$4x - 14y = -2$$

Subtract the two equations.

$$6y = 15$$

Solve for y.

$$y = \frac{15}{6}$$

Solve for x.

$$4x - 8\left(\frac{15}{6}\right) = 13$$
$$4x = 33$$
$$x = \frac{33}{4}$$

Find $2x - y$.

$$2\left(\frac{33}{4}\right) - \frac{15}{6} = 14$$

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Notice that if I subtract the second equation from the first equation, I get $2x - y$!

$$4x - 8y = 13$$
$$2x - 7y = -1$$

$$2x - y = 13 - (-1) = 14$$

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Geometry

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<tr><td colspan="2">A circle with a radius of 6 inches has a sector with a central angle of $x°$ and an area of 15π square inches. What is the area of a sector with the same central angle but with a radius of 12 inches?</td></tr>
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$$\frac{x°}{360} \cdot \pi 6^2 = 15\pi$$

$$\frac{x°}{360} = \frac{5}{12}$$

$$12x = 5(360)$$

$$x = \frac{1800}{12} = 150°$$

$$\frac{150}{360} \cdot \pi 12^2 = 60\pi$$

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I know that to find the area of a sector I take a fraction of the total area of the circle. I took some fraction of the circle's area to get the area for the first sector. In the second problem, the only thing that changes is the area of the circle. We're taking the same fraction of something larger. How much larger? 4x larger, since the radius which is twice as large as it used to be is squared. I can safely ignore what the actual angle measure is; whatever portion I have of the first circle will be the portion I have of the second circle. The areas are proportional. So, the area of the new sector should be four times the area of the original sector and $15\pi(4) = 60\pi$ *square inches.*

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Precalculus

Solve $e^{2x} - 3e^{x} - 40 = 0$	
Non-structural thinking:	Structural thinking:
I've never seen one like this before. I have no idea how to solve this!	I've never seen one like this before but it kind of looks like a quadratic. I see $(\text{something})^2 - 3(\text{something}) - 40$. Can I factor this? $$(e^x - 8)(e^x + 5) = 0$$ I can use the zero product property here. What would make $e^x = 8$? If $x = \ln 8$! What would make $e^x = -5$? Wait! The graph of $y = e^x$ is always above the x-axis! It will never touch $y = -5$. There must only be one solution to this equation.